Poisson Process Driven Stochastic Differential Equations for Bivariate Heavy Tailed Distributions

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***Abstract*— Stochastic differential equations have been used intensively in stochastic control. In this paper, we present 2-dimensional Poisson Counter Driven Stochastic Differential Equation (PCSDE) models that lead to correlated bivariated power law behaviors. We propose two types of 2D PCSDE models and study their tail dependence behavior. The first model generates tail dependence coefficient with values either 0 or 1; while the second model could have the values between 0 and 1. We discuss plausible application of our models in complex network generative models.**

I. Introduction

Power-law distributions have been observed for a variety of phenomena [1], including word frequency of English and other languages, size of earthquakes, firing pattern in neural networks, wealth distribution, population of cities, etc. The ubiquity of power-law distributions have motivated researchers to search for mechanisms to explain their origins. In [2], the author described several generative models for power law distributions, including preferential attachment, optimization, multiplicative models, and so forth. [3] attempts a universal mechanism by showing that power-law emerges when an exponentially growing process is stopped at exponentially distributed random time. In [4], Jiang *et al.* presented different Stochastic Differential Equation (SDE) models in this spirit. The PCSDE models in this paper can be used to generate power law at lower tail, upper tail, or near a critical point.

In this work, we start with a 1D PCSDE model for upper tail power law behavior [4] and then explore a bivariate extension. Such extensions of the PCSDE model are very useful since 2D power-law distributions have been found in some directed real networks, such as some citation networks (arXiv, CiteSeer, US patent), and some social networks (Youtube, Flickr, Livejournal) [5]. Comparing to the citation networks, strong correlations are shown between the nodes' in-degree and out-degree in social networks, as shown in Fig. 1. In this work, we study the generative mechanism for the correlations between the observed data. In [6], Asimit *et al*. proposed a new type of multivariate Pareto distributions. The new distribution has arbitrarily parameterized margins comparing to the traditional multivariate Pareto distribution of the second kind [7]. Meanwhile, the new distribution in [6] is not differentiable everywhere and this phenomenon is also observed in some real datasets with correlations, like in the case of Youtube (as shown in Figure 1(b)). The first a 2D PCSDE model with a shared Poisson counter [8] is formulated on the basis of the bivariate power-law distribution proposed by Asimit *et al*. This model generates correlated bivariate power law distributions.

The tail behavior of a bivariate model can be studied by computing the 'tail dependence coefficient', which is defined as $\lim_{x\to\infty} P(X_2 > x|X_1 > x)$. The tail dependence coefficient relates to the dependence among the extreme values and can be used to predict how the system evolves in the future. The 2D PCSDE model with a shared Poisson counter, however, has the limitation that the tail dependence coefficient is always zero, regardless of the parameters. In this work, we propose two types of modifications to allow nonzero tail dependence coefficient. The first model introduces a Markov on-off process to modulate the Poisson counters so that the active sessions alternate between the independent and the shared Poisson counters. Here the two growth processes are uncoupled within each session, but their life times, which measure the time between occurrence of the last jump of the Poisson counter and the observation, are correlated. The second model takes a complementary approach: the two growth processes are coupled, but their life times are independent.

In our point of view, the models in this work provide possible explanations to the dependence observed among large values in real data. With some further modifications, such as adding Brownian motion part, PCSDE models can be applied to fit real data and predict behaviors under extreme values.

The rest of the paper is organized as follows. Section II starts with a PCSDE model for upper tail power law behavior. Then, a 2D extension with a shared Poisson counter is given and the tail behavior is analyzed. The two modified models are presented in Sections III and IV, respectively. In Section V, we show applications of our models. The 1D model in Section II and 2D model in Section III are connected to the evolution of undirected and directed complex network generative models. Section VI concludes the paper.

II. 1D PCSDE Model and 2D extension

In this section, we review a 1D PCSDE model for upper tail power law distribution and its 2D extension with a shared Poisson counter.

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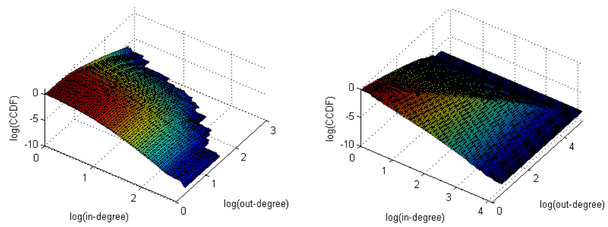


(a) US patent (b) Youtube

Fig. 1. 2D power-law data in citation network and social network.

A. 1D PCSDE Model

A simple PCSDE model that produces an upper tail power-law distribution in steady state is as follows [4],

$$dX(t) = \beta X(t)dt + (x_0 - X(t-))dN(t), \tag{1}$$

where $\beta, x_0 > 0$. N is a Poisson process with rate λ. In this model, X grows exponentially with rate β, and reverts to x_0 after an exponential distributed life time with rate λ. The whole process then repeats.

We give the corresponding characteristic function $\Phi_X(k,t) = \mathrm{E}[e^{jkX(t)}]$ by Ito's rule,

$$(\frac{\partial}{\partial t} - \beta k \frac{\partial}{\partial k})\Phi_X(k,t) = -\lambda \Phi_X(k,t) + \lambda e^{jkx_0}. \tag{2}$$

Solve the Equation (2) as in [4],

$$\begin{aligned}\Phi_X(k,t) = &e^{-\lambda t}\Phi_X(ke^{\beta t},0)\\ &+\lambda\int_0^t e^{-\lambda(t-s)} e^{jx_0 k e^{\beta(t-s)}}ds.\end{aligned} \tag{3}$$

Change variable with $x = x_0 e^{\beta(t-s)}$ and let $t \to \infty$,

$$\Phi_X(k,\infty) = \frac{\lambda}{\beta x_0}\int_{x_0}^{\infty}\left(\frac{x}{x_0}\right)^{-\frac{\lambda}{\beta}-1} e^{jkx}dx. \tag{4}$$

The steady-state density of X gives by taking the inverse Fourier transform,

$$f_X(x) = \frac{\lambda}{\beta x_0}\left(\frac{x}{x_0}\right)^{-\frac{\lambda}{\beta}-1}, \ x \geq x_0, \tag{5}$$

and the Complementary Cumulative Distribution Function (CCDF) is,

$$\bar{F}_X(x) = \left(\frac{x}{x_0}\right)^{-\frac{\lambda}{\beta}}, \quad x \geq x_0. \tag{6}$$

B. 2D PCSDE Model

A 2D extension of the 1D model in Section II-A with a shared Poisson Counter [8] is in (7). The model is simplified by letting $\beta_1 = \beta_2 = 1$ and the initial values to be 1.

$$dX_i(t) = X_i(t)dt + (1 - X_i(t-))(dN_0(t) + dN_i(t)), \tag{7}$$

where $i = 1, 2$. N_0, N_1, and N_2 are independent Poisson counters with rates λ_0, λ_1, and λ_2. The marginal steady-state density is computed,

$$f_{X_i}(x_i) = (\lambda_0 + \lambda_i)x_i^{-(\lambda_0+\lambda_i+1)}, \ x_i \geq 1; \tag{8}$$

and the marginal CCDF is

$$\bar{F}_{X_i}(x_i) = x_i^{-(\lambda_0+\lambda_i)}, x_i \geq 1, \ i = 1,2. \tag{9}$$

For the joint case, let

$$\begin{aligned}\Phi(k_1,k_2,t) &= \mathrm{E}[e^{j\sum_i k_i X_i(t)}],\\ \Phi_i(k_i,t) &= \mathrm{E}[e^{jk_iX_i(t)}].\end{aligned}$$

Applying Ito's rule yields:

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - k_i\frac{\partial}{\partial k_i}\right)\Phi\\ &= -\lambda_+\Phi + \lambda_0 e^{j\sum_i k_i} + \lambda_1 e^{jk_1}\Phi_2 + \lambda_2 e^{jk_2}\Phi_1,\end{aligned} \tag{10}$$

where $\lambda_+ = \lambda_0 + \lambda_1 + \lambda_2$. Solve the equation (10) and let $t \to \infty$, we have

$$\begin{aligned}\Phi(k_1,k_2,\infty) = &\int_1^\infty x^{-\lambda_+-1}\lambda_0 e^{j\sum_i k_i x}dx\\ &+\int_1^\infty x_1^{-\lambda_+-1}\lambda_1 e^{jk_1x_1}\Phi_2(k_2x_1,\infty)dx_1\\ &+\int_1^\infty x_2^{-\lambda_+-1}\lambda_2 e^{jk_2x_2}\Phi_1(k_1x_2,\infty)dx_2,\end{aligned} \tag{11}$$

and the inverse Fourier transform gives

$$\begin{aligned}f_{X_1,X_2}(x_1,x_2) = &\lambda_0 x_1^{-\lambda_+-1}u(x_1-1)\delta(x_1-x_2)\\ &+\lambda_1 x_1^{-\lambda_+-1} f_{X_2}(x_2x_1^{-1})x_1^{-1}u(x_1-1)\\ &+\lambda_2 x_2^{-\lambda_+-1} f_{X_1}(x_1x_2^{-1})x_2^{-1}u(x_2-1),\end{aligned} \tag{12}$$

where $u(x) = 1$ when $x \geq 0$; otherwise, $u(x) = 0$; $\delta(x)$ is the Dirac delta function. The two variables in this model are not independent since $f_{X_1,X_2}(x_1,x_2) \neq f_{X_1}(x_1)f_{X_2}(x_2)$.

The joint CCDF of the model is computed from (12),

$$\begin{aligned}&\bar{F}_{X_1,X_2}(x,x)\\ =&\int_x^\infty dx_1 \int_x^\infty dx_2 f_{X_1,X_2}(x_1,x_2)\\ =&\lambda_0\int_x^\infty dx_1 x_1^{-\lambda_+-1}\\ &+\lambda_1\int_x^\infty dx_1 x_1^{-\lambda_+-1}\int_{x_1}^\infty dx_2 f_{X_2}(x_2x_1^{-1})x_1^{-1}\\ &+\lambda_2\int_x^\infty dx_2 x_2^{-\lambda_+-1}\int_{x_2}^\infty dx_1 f_{X_1}(x_1x_2^{-1})x_2^{-1}\\ =&x^{-\lambda_+}.\end{aligned} \tag{13}$$

With the marginal CCDF in (9), we have the tail dependence coefficient of this model,

$$P(X_2 > x | X_1 > x) = \frac{\bar{F}_{X_1,X_2}(x,x)}{\bar{F}_{X_1}(x)} = x^{-\lambda_2} \xrightarrow{x\to\infty} 0. \tag{14}$$

As indicated in (14), although the model in (7) is useful in generating correlated 2D power-law data, the tail dependence coefficient is always 0. In the next two sections, we pursue modifications to the model in (7) to produce nonzero tail dependence coefficient.

III. Modulated 2D PCSDE Model of Type I

The model in (7) is asymptotically independent due to the existence of two independent Poisson counters. In our first modulated model, we consider shutting down the two independent Poisson counters occasionally.

A. 2D Models with Markov On-off Modulation

Define a Markov on-off process,

$$dY(t) = (1 - Y(t))dM_1(t) - YdM_2(t) \tag{15}$$

where M_1 and M_2 are independent Poisson counters with rate μ_1 and μ_2. Our modified 2D PCSDE model as follows,

$$dX_i(t) = X_i(t)dt + (1 - X_i(t-)) \tag{16}$$
$$\cdot ((1 - Y(t))dN_0(t) + Y(t)dN_i(t)), \tag{17}$$

where $i = 1, 2$. Thus, the two independent Poisson counters are effective when the Markov on-off process is "on" and the shared Poisson counter N_0 is effective when the Markov on-off process is "off". Define

$$\Phi(k_1, k_2, t) = \mathrm{E}[e^{j\sum_i k_i X_i(t)}],\ \Phi_i(k_i, t) = \mathrm{E}[e^{jk_i X_i(t)}],$$
$$\Psi(k_1, k_2, t) = \mathrm{E}[Y(t)e^{j\sum_i k_i X_i(t)}],\ \Psi_i(k_i, t) = \mathrm{E}[Y(t)e^{jk_i X_i(t)}],$$

and let $m(t) = \mathrm{E}[Y(t)]$. For the marginal, Ito's rule yields:

$$\left(\frac{\partial}{\partial t} - k_i \frac{\partial}{\partial k_i}\right) H_i = -A_i H_i + b_i e^{jk_i}, \tag{18}$$

where

$$H_i = \begin{pmatrix} \Phi_i \\ \Psi_i \end{pmatrix}, \quad A_i = \begin{pmatrix} \lambda_0 & \lambda_i - \lambda_0 \\ -\mu_1 & \lambda_i + \mu_1 + \mu_2 \end{pmatrix},$$

and

$$b_i = \begin{pmatrix} \lambda_0(1 - m(\infty)) + \lambda_i m(t) \\ \lambda_i m(t) \end{pmatrix}.$$

Equation (18) can be solved as

$$H_i(k_i, t) = e^{-A_i t} H_i(k_i e^t, 0) + \int_0^t e^{-A_i(t-s)} b_i(s) e^{jk_i e^{t-s}} ds. \tag{19}$$

Changing variables by letting $x_i = e^{t-s}$,

$$H_i(k_i, t) = e^{-A_i t} H_i(k_i e^t, 0) + \int_1^{e^t} e^{-A_i \log x_i} b_i(t - \log x_i) e^{jk_i x_i} x_i^{-1} dx_i. \tag{20}$$

With $t \to \infty$,

$$H_i(k_i, \infty) = \int_1^\infty e^{-A_i \log x_i} b_i(\infty) e^{jk_i x_i} x_i^{-1} dx_i. \tag{21}$$

Taking inverse Fourier transform, the marginal steady-state density can be computed as

$$f_{X_i}(x_i) = a x_i^{-A_i} b_i(\infty) x_i^{-1},\ x_i > 1 \tag{22}$$

where $a = (1, 0)$. Let $\gamma = (1, m(\infty))^T = A_i^{-1} b_i(\infty)$, the marginal CCDF can be computed as

$$\begin{aligned} \bar{F}_{X_i}(x) &= \int_x^\infty a x_i^{-A_i} b_i(\infty) x_i^{-1} dx_i \\ &= a x^{-A_i} A_i^{-1} b_i(\infty), \\ &= a x^{-A_i} \gamma. \end{aligned} \tag{23}$$

For the joint case, Ito's rule yields:

$$\left(\frac{\partial}{\partial t} - \sum_i k_i \frac{\partial}{\partial k_i}\right) H = -AH + be^{j\sum_i k_i} + c\lambda_1 e^{jk_1}\Psi_2 + c\lambda_2 e^{jk_2}\Psi_1. \tag{24}$$

where

$$H = \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}, \quad A = \begin{pmatrix} \lambda_0 & \sum_{i=1,2} \lambda_i - \lambda_0 \\ -\mu_1 & \sum_{i=1,2}(\lambda_i + \mu_i) \end{pmatrix},$$

and

$$b = \begin{pmatrix} 1 - m(t) \\ 0 \end{pmatrix} \lambda_0, \quad c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The solution to equation (24) at $t \to \infty$ is

$$\begin{aligned} &H(k_1, k_2, \infty) \\ &= \int_1^\infty dx e^{-A\log x} b(\infty) e^{j\sum_i k_i x} x^{-1} \\ &+ \int_1^\infty dx_1 e^{-A\log x_1} c\lambda_1 e^{jk_1 x_1} \Psi_2(k_2 x_1, \infty) x_1^{-1} \\ &+ \int_1^\infty dx_2 e^{-A\log x_2} c\lambda_2 e^{jk_2 x_2} \Psi_1(k_1 x_2, \infty) x_2^{-1}. \end{aligned} \tag{25}$$

Taking inverse Fourier transform, we have the joint density

$$\begin{aligned} &f_{X_1,X_2}(x_1, x_2) \\ =& a x_1^{-A} b(\infty) x_1^{-1} u(x_1 - 1)\delta(x_1 - x_2) \\ +& a x_1^{-A} c\lambda_1 f_{X_2}(x_2 x_1^{-1}) m(\infty) x_1^{-2} u(x_1 - 1) \\ +& a x_2^{-A} c\lambda_2 f_{X_1}(x_1 x_2^{-1}) m(\infty) x_2^{-2} u(x_2 - 1). \end{aligned} \tag{26}$$

Let $\gamma = (1, m(\infty))^T = A^{-1}[b(\infty) + \lambda_1 c m(\infty) + \lambda_2 c m(\infty)]$, we have

$$\begin{aligned} &\bar{F}_{X_1 X_2}(x, x) \\ =& \int_x^\infty dx_1 a x_1^{-A} b(\infty) x_1^{-1} \\ +& \int_x^\infty dx_1 a x_1^{-A} c\lambda_1 x_1^{-1} \int_{x_1}^\infty dx_2 f_{X_2}(x_2 x_1^{-1}) m(\infty) x_1^{-1} \\ +& \int_x^\infty dx_2 a x_2^{-A} c\lambda_2 x_2^{-1} \int_{x_2}^\infty dx_1 f_{X_1}(x_1 x_2^{-1}) m(\infty) x_2^{-1} \\ =& a x^{-A} A^{-1}[b(\infty) + \lambda_1 c m(\infty) + \lambda_2 c m(\infty)] \\ =& a x^{-A} \gamma. \end{aligned} \tag{27}$$

Let $\xi_\pm^i$ be the eigenvalues of A_i and $\xi_\pm$ be the eigenvalues of A, we have

$$\xi_\pm^{(1)} = \frac{\lambda_0 + \lambda_1 + \mu_1 + \mu_2}{2} \pm \frac{\sqrt{(\lambda_1 - \lambda_0 + \mu_2 - \mu_1)^2 + 4\mu_1\mu_2}}{2}, \tag{28}$$

and

$$\xi_{\pm} = \frac{\lambda_0+\lambda_1+\lambda_2+\mu_1+\mu_2}{2} \pm \frac{\sqrt{(\lambda_1+\lambda_2-\lambda_0+\mu_2-\mu_1)^2+4\mu_1\mu_2}}{2}. \quad (29)$$

It is easy to check that $\xi_- - \xi_-^{(1)} > 0$, which implies

$$P(X_2 > x | X_1 > x) \sim Cx^{-(\xi_- - \xi_-^{(1)})} \xrightarrow{x\to\infty} 0. \quad (30)$$

As indicated in (30), this model is still asymptotically independent.

B. Modulated Model with 'Manually Reverting'

In this subsection, we consider manually reverting the variables to their initial values whenever the Markov on-off process changes its state. Thus, for any individual growth process between two successive reverting, the Markov on-off process is either in "on" or "off" state during the whole period. The new model is as follows:

$$\begin{aligned} dX_i =& X_i dt + (1-X_{i-}) \\ &\cdot \left((1-Y)(dN_0+dM_1) + Y(dN_i+dM_2)\right), \end{aligned} \quad (31)$$

where $i=1,2$. We omit the t in parentheses in this and the following equations. Use the same method in Section III-A, we have

$$\bar{F}_{X_i}(x) = ax^{-A_i}\gamma \quad \bar{F}_{X_1,X_2}(x,x) = ax^{-A}\gamma, \quad (32)$$

where in this model,

$$A_i = \begin{pmatrix} \lambda_0+\mu_1 & \lambda_i - \lambda_0 + \mu_2 - \mu_1 \\ 0 & \lambda_i + \mu_2 \end{pmatrix},$$

and

$$A = \begin{pmatrix} \lambda_0+\mu_1 & \sum_{i=1,2}\lambda_i - \lambda_0 + \mu_2 - \mu_1 \\ 0 & \sum_{i=1,2}\lambda_i + \mu_2 \end{pmatrix}.$$

a and γ are the same as in Section III-A.

Let $\lambda_1 = \lambda_2 \triangleq \lambda$ and do the eigen-decomposition to A_i and A. The marginal and joint CCDF of this model become:

$$\bar{F}_{X_i}(x) = x^{-(\lambda+\mu_2)}m(\infty) + x^{-(\lambda_0+\mu_1)}(1-m(\infty)), \quad (33)$$

and

$$\bar{F}_{X_1,X_2}(x,x) = x^{-(2\lambda+\mu_2)}m(\infty) + x^{-(\lambda_0+\mu_1)}(1-m(\infty)), \quad (34)$$

Denote $\Delta\mu = \mu_1 - \mu_2$. The tail dependence coefficient of this model is

$$\lim_{x\to\infty} P(X_2 > x | X_1 > x) = \begin{cases} 1 & \lambda > \lambda_0 + \Delta\mu \\ \frac{\mu_2}{\mu_1+\mu_2} & \lambda = \lambda_0 + \Delta\mu \\ 0 & \lambda < \lambda_0 + \Delta\mu. \end{cases} \quad (35)$$

Modulated model in this subsection successfully generates nonzero tail dependence coefficient. However, it seems that the case when fractional tail dependence coefficient appears is unstable. When a parameter is perturbed, the coefficient goes to 1 or 0. In the next Section, we propose a model which can generate fractional tail dependence coefficients in a more natural way.

IV. Modulated 2D PCSDE Model of Type II

We propose the following model where the growth processes of the two variables are coupled,

$$d\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} dt + \begin{pmatrix} 1-X_{1-} \\ 0 \end{pmatrix} dN_1 + \begin{pmatrix} 0 \\ 1-X_{2-} \end{pmatrix} dN_2. \quad (36)$$

For the marginal tail, we prove that the stationary state of variables X_1 and X_2 satisfy the following stochastic recursion,

$$X \stackrel{d}{=} AX + B, \quad (37)$$

where A is defined in Equation (38). Denote the rates of Poisson counters N_1 and N_2 to be $\lambda_1 = \lambda_2 \triangleq \lambda$. Let (T_j) be i.i.d. $\exp(2\lambda)$ random variables, independent of a $Ge(1/2)$ random variable N. Then:

$$A = \begin{cases} \frac{e^{T_1(1+\beta)}+e^{T_1(1-\beta)}}{2} & N=0 \\ \frac{e^{T_1(1+\beta)}-e^{T_1(1-\beta)}}{2} \cdot \frac{e^{T_2(1+\beta)}-e^{T_2(1-\beta)}}{2} & \\ \cdot \prod_{j=3}^{N+1} \frac{e^{T_j(1+\beta)}+e^{T_j(1-\beta)}}{2} & N \ge 0 \end{cases} \quad (38)$$

With equation (37), we have

$$P(X > x) \sim Cx^{-\alpha}, \ x\to\infty, \quad (39)$$

where $\alpha > 0$ is such that $\mathrm{E}[A^\alpha] = 1$. Let

$$\begin{aligned} I_1 &= \mathrm{E}\left[\left(\frac{e^{T_1(1+\beta)}+e^{T_1(1-\beta)}}{2}\right)^\alpha\right] \\ &= \int_0^\infty 2\lambda e^{-2\lambda t}\left(\frac{e^{t(1+\beta)}+e^{t(1-\beta)}}{2}\right)^\alpha dt, \\ I_2 &= \mathrm{E}\left[\left(\frac{e^{T_1(1+\beta)}-e^{T_1(1-\beta)}}{2}\right)^\alpha\right]. \\ &= \int_0^\infty 2\lambda e^{-2\lambda t}\left(\frac{e^{t(1+\beta)}-e^{t(1-\beta)}}{2}\right)^\alpha dt. \end{aligned} \quad (40)$$

We have

$$\mathrm{E}[A^\alpha] = \frac{1}{2}I_1 + I_2^2\sum_{n=1}^\infty \frac{1}{2^{n+1}}I_1^{n-1}. \quad (41)$$

Equation (41) converges to

$$\mathrm{E}[A^\alpha] = \frac{1}{2}I_1 + I_2^2\frac{1}{4-2I_1} \quad (42)$$

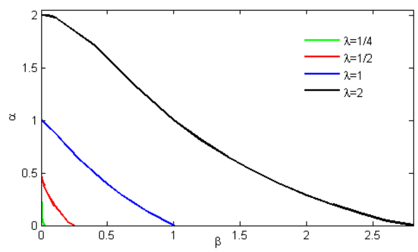
with $I_1 < 2$. By changing the variables, we have

$$I_1 = \frac{\lambda 2^{-\alpha}}{\beta}\int_0^1 z^{\frac{2\lambda-\alpha(1+\beta)}{2\beta}-1}(1+z)^\alpha dz, \quad (43)$$

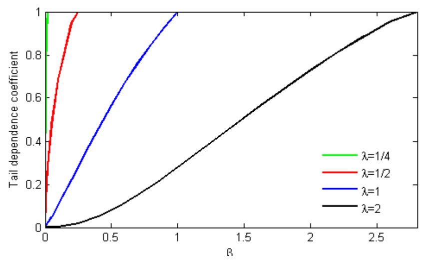
and

$$I_2 = \frac{\lambda 2^{-\alpha}}{\beta}B\left(\frac{2\lambda-\alpha(1+\beta)}{2\beta}, \alpha+1\right), \quad (44)$$

where $B(x,y)$ is the beta function. We use MATLAB to approximate the integration in (43) numerically and solve α with a given β.



(a) α value as a function of β



(b) Tail dependence coefficient as a function of β

Fig. 2. Numerical results of α and tail dependence coefficient as a function of β, $\lambda = 1/4, 1/2, 1,$ and 2.

For the joint case, let T be an $\exp(2\lambda)$ random variable. Given $T = t$, $u \sim \mathbb{U}(0, t)$. Denote

$$V = \frac{e^{u(1+\beta)} - e^{u(1-\beta)}}{2};$$
$$W = \frac{e^{u(1+\beta)} + e^{u(1-\beta)}}{2}. \quad (45)$$

In the stationary regime, we have:

$$(X_1, X_2) \overset{d}{=} \begin{cases} (XV + W, XW + V) & w.p. \ \frac{1}{2} \\ (XW + V, XV + W) & w.p. \ \frac{1}{2} \end{cases}, \quad (46)$$

where X is the same as in Equation (39).

The tail dependence coefficient of this model can be computed by Breiman's lemma [9],

$$\lim_{x\to\infty} P(X_2 > x | X_1 > x) = \frac{2\mathrm{E}[V^\alpha]}{\mathrm{E}[V^\alpha] + \mathrm{E}[W^\alpha]}. \quad (47)$$

EV^α and EW^α can be estimated by generating i.i.d. samples from distribution in equation (45) and computing the sample mean. Since $0 < V < W$ for any $\beta > 0$, the tail dependence coefficient of this model is between 0 and 1.

Let $\lambda = 1/4, 1/2, 1$ and 2, the numerical results are plotted in Fig. 2. As we know, when $\beta = 0$, the tail exponent $\alpha = \lambda$ and the tail dependence coefficient is 0. As shown in Fig. 2, α decreases as β increasing, which means the tail becomes heavier. Meanwhile, the tail dependence coefficient increases with the increasing of β value and reaches 1 at the same time when α reaches 0. For larger Poisson rate λ, the critical β value that makes the tail exponent α approach 0 is also larger.

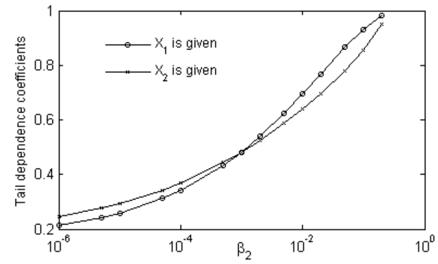


Fig. 3. Tail dependence coefficients as a function of β_2 ($\lambda = 0.25, \beta_1 = 0.001$)

Consider the model in a more general form:

$$d\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & \beta_1 \\ \beta_2 & 1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} dt + \begin{pmatrix} 1 - X_{1-} \\ 0 \end{pmatrix} dN_1 + \begin{pmatrix} 0 \\ 1 - X_{2-} \end{pmatrix} dN_2. \quad (48)$$

Use the same method, we prove X_1 and X_2 both satisfy the stochastic recursion in (37), where A is the same as in (38) with $\beta = \sqrt{\beta_1\beta_2}$.

For the joint case, let $V_1 = \sqrt{\frac{\beta_1}{\beta_2}}V$ and $V_2 = \sqrt{\frac{\beta_2}{\beta_1}}V$, where V is the same as in Equation (45). Then we have

$$(X_1, X_2) \overset{d}{=} \begin{cases} (XV_1 + W, XW + V_2) & w.p. \ \frac{1}{2} \\ (XW + V_1, XV_2 + W) & w.p. \ \frac{1}{2} \end{cases}. \quad (49)$$

and the tail dependence coefficients become

$$\lim_{x\to\infty} \frac{P(X_1 > x, X_2 > x)}{P(X_i > x)} = \frac{\mathrm{E}\left[\min\left(V_1, W\right)^\alpha\right] + \mathrm{E}\left[\min\left(W, V_2\right)^\alpha\right]}{\mathrm{E}[V_i^\alpha] + \mathrm{E}[W^\alpha]}, \ i = 1, 2. \quad (50)$$

This model generates different tail dependence coefficients with X_1 or X_2 given. Let $\lambda = 1/4$, fix the value $\beta_1 = 0.001$ and increase β_2 value to compute the tail dependence coefficients in (50). As shown in Fig. 3, $\lim_{x\to\infty} P(X_2 > x | X_1 > x) < \lim_{x\to\infty} P(X_1 > x | X_2 > x)$ when $\beta_1 > \beta_2$ and vice versa.

To get a whole picture of the joint distribution of the second modified mode, we generate samples pairs of (X_1, X_2) in (46), and draw the CCDF of the samples. With $\lambda = 2$, $\beta = 0.2$, we have $\alpha = 1.9203$. $100,000$ samples are generated and the CCDF is shown in Fig. 4. As we can see, the CCDF has a shape of a saddle on top and this feature actually is not discovered in 2D power-law data we know.

V. PCSDE Model for Complex Networks

The PCSDE models we proposed can be used to explain power law data in complex networks potentially. In the network generative algorithm with preferential attachment, the network grows by adding new nodes and new edges in each step. Preferential attachment means, when adding new edges, the target node is selected with a probability proportional to the node's current degree. For directed graphs, the node is

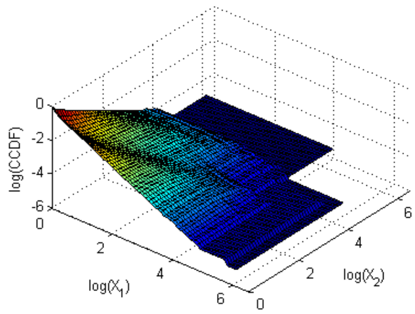


Fig. 4. CCDF of $N = 100,000$ i.i.d. samples of (X_1, X_2) in (46).

selected according to the node's current in-degree or out-degree.

Let EM be the expectation of the total degree (in-degree or out-degree for directed graph) added in each step, with (1) EM_1: the degree associated with the new nodes; and (2) EM_2: the degree associated with the existing nodes in the original graph. Assume the number of nodes in the network grows exponentially with rate λ. Then the life time of the nodes, which is defined as the time period between the instant when the node is born and the observation instant, follows exponential distribution with rate λ. Then with preferential attachment, we prove that, the expected degree that a node gets in the network also grows exponentially with time, and the rate β has the following relationship to λ,

$$\beta = \frac{EM_2}{EM}\lambda. \tag{51}$$

So, in the generative models, expected degree grows exponentially, while the life time follows exponential distribution, which means the PCSDE model in (1) provides an interpretation to the power-law behavior in some network generative models with preferential attachment. For B-A model in [10], the algorithm attaches half of the undirected edges to the new node and half to the nodes in the existing network in each step. With $EM_2 = \frac{1}{2}EM$, we have $\beta = \frac{1}{2}\lambda$ for B-A model.

While univariate PCSDE models are useful, 2D PCSDE model can also be used to model the expected in- and out-degree growths in directed networks, such as the Bollobás model in [11]. In this model, with probability p one appends a new node with a directed edge (with p_1, introducing in a new in-degree; with p_2, introducing in a new out-degree); with probability $q = 1-p$ one appends a directed edge to the existing graph with preferential attachment. Each new node is given a bias to in-degree ϵ_{in} and a bias to out-degree ϵ_{out} to avoid 0 initial in- and out-degree. We give the following 2D PCSDE model,

$$\begin{aligned} dX_1 &= \beta_1 X_1 dt + (1+\epsilon_{\text{in}} - X_1)dN_1 + (\epsilon_{\text{in}} - X_1)dN_2 \\ dX_2 &= \beta_2 X_2 dt + (\epsilon_{\text{out}} - X_2)dN_1 + (1+\epsilon_{\text{out}} - X_2)dN_2, \end{aligned} \tag{52}$$

with Poisson rates $\lambda_1 + \lambda_2 = \lambda$ and $\lambda_1/\lambda_2 = p_1/p_2$. The expected in-degree added in each step is $EM^{in} = 1 + p\epsilon_{in}$ and the expected in-degree added with preferential attachment is $EM_2^{in} = p_1 + q$. So, the growth rate of expected in-degree β_1 has the following relationship to λ: $\beta_1 = \frac{p_1+q}{1+p\epsilon_{in}}\lambda$. Similarly, we have $EM^{out} = 1 + p\epsilon_{out}$ and $EM_2^{out} = p_2 + q$, which gives $\beta_2 = \frac{p_2+q}{1+p\epsilon_{out}}\lambda$.

By adding Brownian motion component into the PCSDE model, our stochastic model may be used to describe degree growth of a single randomly selected node in complex networks, like the model below:

$$dX = \beta X dt + g(X)dW + (x_0 - X_-)dN, \tag{53}$$

where W is a standard Brownian motion. The model with $g(X) = \sigma X$ is already discussed in [4], which produces double Pareto distribution. The 2D extension of the SDE model with both Browian motion and Poisson counter, thus could be used to describe the in-degree and out-degree growth of a single node in a directed network. This part contains in our further work.

VI. Conclusions

We develop two types of 2D PCSDE models to generate correlated bivariate power-law distributions. We study the tail dependence of these models. The results indicate that the model of Type II might be the more interesting one since it generates fractional tail dependence coefficients. However, fractional tail dependence coefficient may not be the case in real data, since this model has a joint distribution that does not fit the body part of existing data. In our model of Type I, a shared Poisson counter implies that the two variables start their growth processes at the same time, which further implies a common cause behind. When the empirical data indicates dependence in the regions of very large values we might want to infer that there is a common cause for the very large values observed.

References

[1] A. Clauset, C.R. Shalizi and M.E.J. Newman. Power-Law Distributions in Empirical Data, SIAM REVIEW. vol. 51, no. 4, pp. 661-703, 2009.
[2] M. Mitzenmacher. A Brief History of Generative Models for Power Law and Lognormal Distributions, Internet Mathematics. vol. 1, no. 20, pp. 226-251, 2004.
[3] W.J. Reed and B.D. Huges. From Gene Families and Genera to Incomes and Internet File Sizes: Why Power-Laws are So Common in Nature, Physical Review E66. 067103, 2002.
[4] B. Jiang, R. Brockett, W. Gong and D. Towsley. Stochastic Differential Eqtations for Power-Law Behavior, 51st IEEE Conference on Decision and Control. pp. 6696-6701, Dec 10-13, 2012.
[5] J. Kunegis, KONECT: the Koblenz Network Collection, WWW 2013 Companion, May 13-17, 2013, Rio de Janeiro, Brazil.
[6] A.V. Asimit, E. Furman and R. Vernic, On a Multivariate Pareto Distribution, Insurance: Mathematics and Economics, vol. 46, issue 2, pp. 308-316, 2010.
[7] B.C. Arnold, Pareto Distributions, International Co-operative Publishing House. ISBN 0-89974-012-X. Chapter 6.
[8] B. Jiang. Talk at University of Massachusetts, Amherst. Amherst, MA, USA, August 21, 2012.
[9] L. Breiman. On Some Limit Theorems Similar to the Arc-Sin Law. Theory Probab. Appl.. vol. 10, no. 2, pp. 323-331, 1965.
[10] AL. Barabási and R. Albert, Emergence of Scaling in Random Networks, Science, vol. 286, issue 5439, pp. 509-512, 1999.
[11] B. Bollobás, C. Borgs, J. Chayes and O. Riordan, Directed Scale-free Graphs, Proceedings of SODA'03, pp.132-139, 2003.