1. ([R], Page 689, Exercise 2(a)(b)(d)) Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?
   a) a, b, e, c, b
   b) a, d, a, d, a
   d) a, b, e, c, b, d, a

   Answer Area:
   a) A simple path. The length is 4.
   b) A circuit, not simple. The lengths is 4.
   d) Not path.

2. ([R], Page 689, 6) How many connected components does each of the graphs in Exercises 3-5 have? For each graph find each of its connected components.

   Answer Area:
3. ([R], Page 690, 14(a)(b)(c)) Find the strongly connected components of each of these graphs.

a) (abe),(d),(c)

b) (a),(f),(b),(cde)

c) (e),(abcdfghi)

4. ([R], Page 692, Exercise 50(a)(c)(d)) For each of these graphs, find $\kappa(G)$, $\lambda(G)$, and $\min_{v \in V} \deg(v)$, and determine which of the two inequalities in $\kappa(G) \leq \lambda(G) \leq \min_{v \in V} \deg(v)$ are strict.
a) $\kappa(G) = 1, \lambda(G) = 2, \min_{v \in V} \deg(v) = 2$, thus $\kappa(G) < \lambda(G) = \min_{v \in V} \deg(v)$.

c) $\kappa(G) = 2, \lambda(G) = 2, \min_{v \in V} \deg(v) = 3$, thus $\kappa(G) = \lambda(G) < \min_{v \in V} \deg(v)$.

d) $\kappa(G) = 4, \lambda(G) = 4, \min_{v \in V} \deg(v) = 4$ (since any three vertices remain are connected), thus $\kappa(G) = \lambda(G) = \min_{v \in V} \deg(v)$.

5. ([R], Page 692, Exercise 52(b)) Show that if $G$ is a connected graph with $n$ vertices then 

b) $\lambda(G) = n - 1$ if and only if $G = K_n$

Note: Please prove without using the inequalities $\kappa(G) \leq \lambda(G) \leq \min_{v \in V}(v)$. You can proceed by induction on the number of vertices.

**Answer Area:**

b) We prove it by induction. $n := 2$, $\lambda(K_2) = 1$. Suppose $n := n$, $\lambda(K_n) = n - 1$. Then for $n := n + 1$, let $a$ be the $(n + 1)$-th vertex in $K_{n+1}$, then just removing $n$ edges from $a$ to the other $n$ vertices disconnects $K_{n+1}$. Next, we prove that $n$ is the minimal number because if we just delete $n - 1$ edges in $K_{n+1}$ randomly, it only ensures that there is a disconnected graph in one $K_n$, and there always exists an additional vertex that connects the whole graph. So we can conclude $\lambda(G) = n - 1$.

6. ([R], Page 692, Exercise 63) Show that a simple graph $G$ is bipartite if and only if it has no circuits with an odd number of edges.

**Hint:** Fix any vertex $v$ and consider partition the vertices into those which can be connected from $v$ through a path of odd length and those which can be connected from $v$ through a path of even length.

**Answer Area:**

⇒ If $G$ is bipartite with vertex sets $V_1$ and $V_2$, every step alone a walk takes you either from $V_1$ to $V_2$ or from $V_2$ to $V_1$. To end up where you started, you must take an even number of steps, so there are no circuits with an odd number of edges.

⇐ Suppose that all circuits have even length. We can assume that the graph is connected, because if it is not, then we can just work on one component at a time. Let $v$ be a vertex of the graph, and let $V_1$ be the set of all vertices to which there is a path of odd length starting at $v$, and let $V_2$ be the set of all vertices to which there is a path of even length starting at $v$. We put $v$ in $V_2$. Because the graph (or the component) is connected, every vertex lies in $V_1$ or $V_2$, but not both. Thus, the set of vertices has been partitioned into two sets. Since all circuits have even length, there is no edge with both endpoints in the same set $V_1$ or $V_2$. 

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7. ([R], Page 704, Exercise 6, 8) In Exercises 6,8 determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

Answer Area:

6) There doesn’t exist a Euler circuit but exists a Euler path since there’re exactly two vertices of odd degree (b and c).

The Euler path is $badejihgcibc$.

8) There exists a Euler circuit.

The Euler circuit is $abghcbdcjediomhnlgfklmfa$.

8. ([R], Page 705, Exercise 36) In Exercises 36 determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

36.
**Answer Area:**

There exists a Hamilton circuit that is aedghi fcba.

9. ([R], Page 705, Exercise 40) Does the graph in Exercise 33 have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.

**Answer Area:**

The graph doesn’t have a Hamilton path since there are three vertices of degree 1 (e, g, f), however if there’re a Hamilton path then there’re at most 2 vertices of degree 1.

10. ([R], Page 706, Exercise 49) Show that there is a Gray code of order $n$ whenever $n$ is a positive integer, or equivalently, show that the $n$-cube $Q_n, n > 1$, always has a Hamilton circuit. [Hint: Use mathematical induction. Show how to produce a Gray code of order $n$ from one of order $n - 1$.]

**Answer Area:**

The result is trivial for $n = 1$: code is 0,1. Assume we have a Gray code of order $n$. Let $c_1, \ldots, c_k (k = 2^n)$ be such a code. Then $0c_1, \ldots, 0c_k, 1c_k, \ldots, 1c_1$ is a Gray code of order $n+1$.

11. ([R], Page 706, Exercise 55) Show that a bipartite graph with an odd number of vertices does not have a Hamilton circuit.

**Answer Area:**

Suppose $G = (V, E)$ is a bipartite graph with $V = V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$ and no edge connects two vertices in $V_1$ or two vertices in $V_2$. Suppose that $G$ has a Hamilton circuit. Such a circuit must be of the form $a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_1$, where $a_i \in V_1, b_i \in V_2$ for $i = 1, 2, \ldots, k$. Because the Hamilton circuit visits each vertex exactly once, except for $a_1$, where it begins and ends, the number of vertices in the graph equals an even number $2k$. Hence, a bipartite graph with an odd number of vertices cannot have a Hamilton circuit.