Exercise Sheet 7
Discrete Mathematics by Hongfei Fu, 2019.10.15

* If there is any problem, please contact TA Peixin Wang (1~8) or Jinyi Wang (9,10).

Note: In the homework, you can rely on Venn diagrams for intuition, but you should write your homework using only formal proofs.

1. ([R], Page 125, Exercise 1(c)) List the members of these sets.
   c) \( \{ x | x \text{ is the square of an integer and } x < 100 \} \)
   
   **Answer Area:**
   \( \{ 0, 1, 4, 9, 16, 25, 36, 49, 64, 81 \} \).

2. ([R], Page 125, Exercise 10(c)(d)) Determine whether these statements are true or false.
   
   c) \( \{ \emptyset \} \in \{ \emptyset \} \)
   d) \( \{ \emptyset \} \in \{ \{ \emptyset \} \} \)
   
   **Answer Area:**
   - False.
   - True.

3. ([R], Page 126, Exercise 17) Suppose that \( A, B, \) and \( C \) are sets such that \( A \subseteq B \) and \( B \subseteq C \). Show that \( A \subseteq C \).
   
   **Answer Area:**
   Suppose \( x \in A \), then by \( A \subseteq B \) we have \( x \in B \), and thus \( x \in C \) since \( B \subseteq C \). So \( A \subseteq C \).

4. ([R], Page 126, Exercise 18) Find two sets \( A \) and \( B \) such that \( A \in B \) and \( A \subseteq B \)
   
   **Answer Area:**
   Let \( A \) be \( \emptyset \) and \( B \) be \( \{ \emptyset \} \), then \( A \in B \) and \( A \subseteq B \).

5. ([R], Page 126, Exercise 45) The defining property of an ordered pair is that two ordered pairs are equal if and only if their first elements are equal and their second elements are equal. Surprisingly, instead of taking the ordered pair as a primitive concept, we can construct ordered pairs using basic notions from set theory. Show that if we define the ordered pair \( (a, b) \) to be \( \{ \{a\}, \{a, b\} \} \), then \( (a, b) = (c, d) \) if and only if \( a = c \) and \( b = d \).
   
   **Hint:** First show that \( \{ \{a\}, \{a, b\} \} = \{ \{c\}, \{c, d\} \} \) if and only if \( a = c \) and \( b = d \).
   
   **Answer Area:**
   We show that \( \{ \{a\}, \{a, b\} \} = \{ \{c\}, \{c, d\} \} \) if and only if \( a = c \) and \( b = d \).
   
   \((\Rightarrow): \)
   
   - **CASE** \( a = b \). Thus \( \{ \{a\}, \{a, b\} \} = \{ \{a\} \} \). Since \( \{ \{a\} \} = \{ \{c\}, \{c, d\} \} \), we have \( a = c = d \), i.e. \( a = c \) and \( b = d \).
   
   - **CASE** \( a \neq b \). Since \( \{ \{a\}, \{a, b\} \} = \{ \{c\}, \{c, d\} \} \), we have \( \{a\} \in \{ \{c\}, \{c, d\} \} \), this implies \( \{a\} = \{c\} \), i.e. \( a = c \). Similarly we have \( \{a, b\} = \{c, d\} \), by \( a = c \), we get \( b = d \).
6. ([R], Page 136, Exercise 24) Let $A$, $B$, and $C$ be sets. Show that $(A - B) - C = (A - C) - (B - C)$.

**Answer Area:**

$(\Rightarrow)$: Suppose $x$ is in the left-hand side. Then $x$ must be in $A$ but in neither $B$ nor $C$. Thus $x \in A - C$, but $x \notin B - C$, so $x \in (A - C) - (B - C)$.

$(\Leftarrow)$: Suppose $x$ is in the right-hand side. Then $x$ must be in $A - C$ and not in $B - C$. These imply that $x \in A$ and $x \notin C$, and also $x \notin B$. Thus we have shown that $x \in (A - B) - C$.

7. ([R], Page 137, Exercise 38(b)) Show that if $A$ and $B$ are sets, then

b) $(A \oplus B) \oplus B = A$

**Note:** The symmetric difference of $A$ and $B$, denoted by $A \oplus B$, is the set containing those elements in either $A$ or $B$, but not in both $A$ and $B$.

**Answer Area:**

$(\Rightarrow)$: Suppose $x \in (A \oplus B) \oplus B$. If $x \in B$, then $x \notin A \oplus B$, which means $x \in A$. If $x \notin B$, then $x \in A \oplus B$, which means $x \in A$.

$(\Leftarrow)$: Suppose $x \in A$. If $x \in B$, then $x \notin A \oplus B$, so it belongs to the left-hand side. If $x \notin B$, then $x \in A \oplus B$, so it also belongs to the left-hand side.

8. ([R], Page 137, Exercise 40) Determine whether the symmetric difference is associative; that is, if $A$, $B$, and $C$ are sets, does it follow that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$?

**Answer Area:**

$(\Rightarrow)$: Suppose $x \in A \oplus (B \oplus C)$, there are two cases: If $x \in A$, then $x \notin B \oplus C$, which implies $x \in B \land x \in C$. So $x \notin A \oplus B$, then we can get that $x \in (A \oplus B) \oplus C$; If $x \notin A$, then $x \in B \oplus C$. When $x \in B \land x \notin C$, $x \in (A \oplus B) \oplus C$. When $x \notin B \land x \in C$, $x \notin A \oplus B$, which means $x \in (A \oplus B) \oplus C$.

$(\Leftarrow)$: It’s similar.

9. ([E], Page 26, Exercise 7(a)) Show that for any sets $A$ and $B$

$$\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$$

**Answer Area:**

Suppose $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$, then $(x \subseteq A) \land (x \subseteq B)$, which means $x \subseteq (A \cap B)$. Thus, $x \in \mathcal{P}(A \cap B)$. We could get $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.

Suppose $x \in \mathcal{P}(A \cap B)$, then $x \subseteq (A \cap B)$, which means $(x \subseteq A) \land (x \subseteq B)$. Thus, $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$. We could get $\mathcal{P}(A) \cap \mathcal{P}(B) \supseteq \mathcal{P}(A \cap B)$.

Then $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$ is proved.
10. ([E], Page 38, Exercise 3) Show that $A \times \bigcup \mathcal{B} = \bigcup \{A \times X | X \in \mathcal{B}\}$

Answer Area:

$$A \times \bigcup \mathcal{B} = \{(a, b) | (a \in A) \land (b \in \bigcup \mathcal{B}) \}$$

$$= \{(a, b) | (a \in A) \land (\exists X \in \mathcal{B}) b \in X \}$$

$$= (\exists X \in \mathcal{B}) \{(a, b) | (a \in A) \land b \in X \}$$

$$= \bigcup \{A \times X | X \in \mathcal{B} \}$$