Connectivity, Paths and Circuits

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Basics of Graph Theory

- graph definitions
- graph terminologies
- special simple graphs
- graph operations
- graph isomorphism
Today’s Topic

- connectivity
- Euler paths and circuits
- Hamilton paths and circuits
Connectivity (连通性)

main textbook, Page 678 – Page 689
Paths in Undirected Graphs

- \( G = (V, E) \): an undirected pseudograph
- \( u, v \in V \)

A path of length \( n \ (n \geq 0) \) from \( u \) to \( v \) in \( G \) is a finite sequence \( e_1, \ldots, e_n \) of edges in \( E \) such that there exists a finite sequence of vertices

\[
\begin{align*}
  u = x_0, & \quad x_1, & \quad \ldots, & \quad x_{n-1}, & \quad x_n = v
\end{align*}
\]

satisfying that the endpoints of each \( e_i \) are \( x_{i-1}, x_i \).
Paths in Undirected Graphs

- $G = (V, E)$: an undirected pseudograph
- $u, v \in V$

A path of length $n$ ($n \geq 0$) from $u$ to $v$ in $G$ is a finite sequence $e_1, \ldots, e_n$ of edges in $E$ such that there exists a finite sequence of vertices

$$u = x_0, x_1, \ldots, x_{n-1}, x_n = v$$

satisfying that the endpoints of each $e_i$ are $x_{i-1}, x_i$.

Remarks

- We write the path $e_1, \ldots, e_n$ as $x_0, x_1, \ldots, x_{n-1}, x_n$ if $G$ is simple.
- If $n = 0$, then the path consists of a single vertex.
Paths in Undirected Graphs

- $G = (V, E)$: an undirected pseudograph
- $u, v \in V$

A path of length $n$ ($n \geq 0$) from $u$ to $v$ in $G$ is a finite sequence $e_1, \ldots, e_n$ of edges in $E$ such that there exists a finite sequence of vertices

$$u = x_0, x_1, \ldots, x_{n-1}, x_n = v$$

satisfying that the endpoints of each $e_i$ are $x_{i-1}, x_i$.

Remarks

- The path $e_1, \ldots, e_n$ is simple if it does not contain duplicate edges.
- The path $e_1, \ldots, e_n$ traverses the edges and passes through the vertices $x_1, \ldots, x_{n-1}$. 
Connectivity

Circuits in Undirected Graphs

- \( G = (V, E) \): an undirected pseudograph
- \( u \in V \)

A path of positive length from \( u \) to \( u \) is a circuit.

Remarks

- The circuit is simple if it does not contain duplicate edges.
- The circuit is said to traverse the edges and pass through the vertices.
Connectivity

### Circuits in Undirected Graphs

- \( G = (V, E) \): an undirected pseudograph
- \( u \in V \)

A path of **positive** length from \( u \) to \( u \) is a **circuit**.

### Remarks

- The circuit is **simple** if it does not contain duplicate edges.
- The circuit is said to **traverse** the edges and **pass through** the vertices.

### Example

![Diagram of a circuit in an undirected graph]

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Paths in Directed Graphs

- $G = (V, E)$: a directed multigraph
- $u, v \in V$:

A path of length $n$ ($n \geq 0$) from $u$ to $v$ in $G$ is a finite sequence $e_1, \ldots, e_n$ of edges in $E$ such that there exists a finite sequence of vertices $u = x_0, x_1, \ldots, x_{n-1}, x_n = v$ satisfying that each $e_i$ starts from $x_{i-1}$ and ends at $x_i$.  

We can define analogously simple paths and circuits.
Connectivity

Paths in Directed Graphs

- $G = (V, E)$: a directed multigraph
- $u, v \in V$:

A path of length $n$ ($n \geq 0$) from $u$ to $v$ in $G$ is a finite sequence $e_1, \ldots, e_n$ of edges in $E$ such that there exists a finite sequence of vertices

$$u = x_0, x_1, \ldots, x_{n-1}, x_n = v$$

satisfying that each $e_i$ starts from $x_{i-1}$ and ends at $x_i$.

We can define analogously . . .

- circuits
- simple paths and circuits
Connectivity

Connectivity: The Undirected Case

- $G = (V, E)$: an undirected pseudograph

Two vertices $u, v \in V$ are connected if there is a path from $u$ to $v$ in $G$. 

Observation
The relation consisting of all ordered pairs of connected vertices is an equivalence relation.

There is a simple path between any two connected vertices.
Connectivity: The Undirected Case

- \( G = (V, E) \): an undirected pseudograph

Two vertices \( u, v \in V \) are \textit{connected} if there is a path from \( u \) to \( v \) in \( G \).

Observation

- The relation consisting of all ordered pairs of connected vertices is an equivalence relation.
- There is a simple path between any two connected vertices.
Connected Graphs

$G = (V, E)$: an undirected pseudograph

We say that $G$ is connected if every pair of vertices in $V$ is connected. Otherwise it is disconnected.
Connectivity

Connected Graphs

- $G = (V, E)$: an undirected pseudograph

We say that $G$ is connected if every pair of vertices in $V$ is connected. Otherwise it is disconnected.

Connected Components

- $G = (V, E)$: an undirected graph

A connected component of $G$ is a connected subgraph that is not a proper subgraph of another connected subgraph of $G$. 
Connected components are "induced subgraphs" of equivalence classes of the connectivity relation.

Example

$G_1$ and $G_2$
Observation: Connected components are "induced subgraphs" of equivalence classes of the connectivity relation.
Connected components are “induced subgraphs” of equivalence classes of the connectivity relation.
Strongly-Connected Directed Graphs

\[ G = (V, E): \text{a directed multigraph} \]

We say that \( G \) is strongly connected if for every pair of vertices \((u, v)\), there is a path from \( u \) to \( v \) and a path from \( v \) to \( u \).
Connectivity

Strongly-Connected Directed Graphs

- $G = (V, E)$: a directed multigraph

We say that $G$ is strongly connected if for every pair of vertices $(u, v)$, there is a path from $u$ to $v$ and a path from $v$ to $u$.

Example

![Graph Diagram]

$\begin{align*}
G = (V, E) & : \text{a directed multigraph} \\
\text{We say that } G \text{ is strongly connected if for every pair of vertices } (u, v), \\
\text{there is a path from } u \text{ to } v \text{ and a path from } v \text{ to } u.
\end{align*}$
Strongly-Connected Components

- \( G = (V, E) \): a directed multigraph

A **strongly-connected component** of \( G \) is a strongly-connected subgraph of \( G \) that is not a proper subgraph of another strongly-connected subgraph of \( G \).
Strongly-Connected Components

- $G = (V, E)$: a directed multigraph

A strongly-connected component of $G$ is a strongly-connected subgraph of $G$ that is not a proper subgraph of another strongly-connected subgraph of $G$.

Example

![Graphs G and H with strongly-connected components](image)
Connectivity

Mutual Reachability

Two vertices $u, v \in V$ are **mutually reachable** if they are paths both from $u$ to $v$ and from $v$ to $u$ in $G$. 

Observation

Mutual reachability is an equivalence relation, and strongly-connected components are "induced subgraphs" of equivalence classes of mutual reachability.
Connectivity

Mutual Reachability

Two vertices $u, v \in V$ are mutually reachable if they are paths both from $u$ to $v$ and from $v$ to $u$ in $G$.

Observation

Mutual reachability is an equivalence relation, and strongly-connected components are “induced subgraphs” of equivalence classes of mutual reachability.
Problem

How well is a graph connected against vertex or edge removal?
Connectivity

Cut Vertices

- $G = (V, E)$: a simple graph

A cut vertex (or articulation point) is a vertex $v \in V$ such that the removal of the vertex $v$ results in more connected components, i.e., the number of connected components in $G - v$ is larger than that in $G$.

Nonseparable Graphs

A connected simple graph is nonseparable if it does not have cut vertices.
Cut Vertices

- $G = (V, E)$: a simple graph

A cut vertex (or articulation point) is a vertex $v \in V$ such that the removal of the vertex $v$ results in more connected components, i.e., the number of connected components in $G - v$ is larger than that in $G$.

Nonseparable Graphs

A connected simple graph is nonseparable if it does not have cut vertices.
Examples

$G_1$

$G_2$

$G_3$

$G_4$
Connectivity

Vertex Cuts

- $G = (V, E)$: a simple graph

A subset $V' \subseteq V$ is called a vertex cut (or separating set) if the subgraph $G - V'$ is disconnected.

Vertex Connectivity $\kappa(G)$

We define the number $\kappa(G)$ to be the minimal number $n$ such that there exists a subset $V' \subseteq V$ that satisfies (i) $|V'| = n$ and (ii) the graph $G - V'$ is either disconnected or a single vertex.
**Vertex Cuts**

- $G = (V, E)$: a simple graph

A subset $V' \subseteq V$ is called a *vertex cut* (or *separating set*) if the subgraph $G - V'$ is disconnected.

**Vertex Connectivity $\kappa(G)$**

We define the number $\kappa(G)$ to be the **minimal** number $n$ such that there exists a subset $V' \subseteq V$ that satisfies (i) $|V'| = n$ and (ii) the graph $G - V'$ is either *disconnected* or a *single vertex*. 
### Remarks

- If $G$ is either disconnected or $K_1$, then $\kappa(G) = 0$.
- If $G$ is either connected with a cut vertex or $K_2$, then $\kappa(G) = 1$.
- $\kappa(K_n) = n - 1$. 
If $G$ is either disconnected or $K_1$, then $\kappa(G) = 0$.
If $G$ is either connected with a cut vertex or $K_2$, then $\kappa(G) = 1$.
$\kappa(K_n) = n - 1$.

If $\kappa(G) \geq k$, then we say that the simple graph $G$ is $k$-vertex-connected.
If $G$ is either disconnected or $K_1$, then $\kappa(G) = 0$.

If $G$ is either connected with a cut vertex or $K_2$, then $\kappa(G) = 1$.

$\kappa(K_n) = n - 1$.

If $\kappa(G) \geq k$, then we say that the simple graph $G$ is $k$-vertex-connected.

If a connected simple graph $G$ contains at least two vertices, then $G$ is 1-connected.

If a nonseparable simple graph $G$ contains at least three vertices, then $G$ is 2-connected (or biconnected).
**Cut Edges**

- $G = (V, E)$: a simple graph

A **cut edge** (or **bridge**) is an edge $e \in E$ such that the removal of the edge $e$ results in more connected components, i.e., the number of connected components in $G - e$ is larger than that in $G$. 
Cut Edges

- $G = (V, E)$: a simple graph

A cut edge (or bridge) is an edge $e \in E$ such that the removal of the edge $e$ results in more connected components, i.e., the number of connected components in $G - e$ is larger than that in $G$.

Edge Cuts

- $G = (V, E)$: a simple graph

A subset $E' \subseteq E$ is called an edge cut if the subgraph $G - E'$ is disconnected.
Edge Connectivity $\lambda(G)$

- $G = (V, E)$: a simple graph with at least two distinct vertices

We define the number $\lambda(G)$ to be the minimal number $n$ of edges such that there exists an edge cut $E' \subseteq E$ that satisfies $|E'| = n$. 

Remarks

If $G$ is disconnected, then $\lambda(G) = 0$.

$\lambda(K_1) := 0$.

$\lambda(G) = |V| - 1$ iff $G$ is a complete graph.
Connectivity

Edge Connectivity $\lambda(G)$

- $G = (V, E)$: a simple graph with at least two distinct vertices

We define the number $\lambda(G)$ to be the minimal number $n$ of edges such that there exists an edge cut $E' \subseteq E$ that satisfies $|E'| = n$.

Remarks

- If $G$ is disconnected, then $\lambda(G) = 0$.
- $\lambda(K_1) := 0$.
- $\lambda(G) = |V| - 1$ iff $G$ is a complete graph.
Theorem

If $G = (V, E)$ is a simple graph with at least two vertices, then we have

$$\kappa(G) \leq \lambda(G) \leq \min_{v \in V} \deg(v).$$

Proof (from [RD], Page 12)

Case 1: There is $v \in V$ that is not incident with edges in $F$. Construct the connected component $C$ of the graph $G - F$ that contains $v$. Then the set of vertices in $C$ that are incident with $F$ is a vertex cut that separates $v$ from $G - C$.

Case 2: Every vertex $v \in V$ is incident with some edge in $F$ and $G$ is not complete. Pick a vertex $v$ such that $\{v\} \cup N(v) \neq V$. Then $N(v)$ is a vertex cut whose size is no more than $|F|$.

Case 3: $G$ is complete. Straightforward.
Theorem

If $G = (V, E)$ is a simple graph with at least two vertices, then we have

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Proof (from [RD], Page 12)

- $\lambda(G) \leq \min_{v \in V} \deg(v)$: Straightforward.
- $\kappa(G) \leq \lambda(G)$: Consider any minimal edge cut $F \subseteq E$. We construct a vertex cut of size no more than $|F|$. 
Theorem

If $G = (V, E)$ is a simple graph with at least two vertices, then we have

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- $\kappa(G) \leq \lambda(G)$: Consider any minimal edge cut $F \subseteq E$. We construct a vertex cut of size no more than $|F|$.
  - **Case 1**: There is $v \in V$ that is not incident with edges in $F$. Construct the connected component $C$ of the graph $G - F$ that contains $v$. Then the set of vertices in $C$ that are incident with $F$ is a vertex cut that separates $v$ from $G - C$. 
Theorem

If \( G = (V, E) \) is a simple graph with at least two vertices, then we have

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\kappa(G) \leq \lambda(G) \leq \min_{v \in V} \deg(v).
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Proof (from [RD], Page 12)

- \( \lambda(G) \leq \min_{v \in V} \deg(v) \): Straightforward.
- \( \kappa(G) \leq \lambda(G) \): Consider any minimal edge cut \( F \subseteq E \). We construct a vertex cut of size no more than \( |F| \).
  - **Case 1**: There is \( v \in V \) that is not incident with edges in \( F \). Construct the connected component \( C \) of the graph \( G - F \) that contains \( v \). Then the set of vertices in \( C \) that are incident with \( F \) is a vertex cut that separates \( v \) from \( G - C \).
  - **Case 2**: Every vertex \( v \in V \) is incident with some edge in \( F \) and \( G \) is not complete. Pick a vertex \( v \) such that \( \{v\} \cup N(v) \neq V \). Then \( N(v) \) is a vertex cut whose size is no more than \( |F| \).
Theorem

If $G = (V, E)$ is a simple graph with at least two vertices, then we have

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- $\kappa(G) \leq \lambda(G)$: Consider any minimal edge cut $F \subseteq E$. We construct a vertex cut of size no more than $|F|$.
  - Case 1: There is $v \in V$ that is not incident with edges in $F$. Construct the connected component $C$ of the graph $G - F$ that contains $v$. Then the set of vertices in $C$ that are incident with $F$ is a vertex cut that separates $v$ from $G - C$.
  - Case 2: Every vertex $v \in V$ is incident with some edge in $F$ and $G$ is not complete. Pick a vertex $v$ such that $\{v\} \cup N(v) \neq V$. Then $N(v)$ is a vertex cut whose size is no more than $|F|$.
  - Case 3: $G$ is complete. Straightforward.
Euler Circuits and Paths

main textbook, Page 693 – 698
Leonhard Euler and the Königsberg’s Seven-Bridge Problem

source: from Wikipedia and the book “Graph Theory” by Reinhard Diestel
Euler Circuits and Paths

Definition

- **G** = (V, E): a directed multigraph

Then we have that

- An **Euler circuit** in **G** is a simple circuit containing every edge of **G**;
- An **Euler path** in **G** is a simple path containing every edge of **G**.
Theorem

A connected undirected pseudograph with at least two vertices has an Euler circuit iff all of its vertices have even degree.
Euler Circuits and Paths

**Theorem**

A connected undirected pseudograph with at least two vertices has an Euler circuit iff all of its vertices have even degree.

**Proof Ideas (see [RD], Page 23)**

- We only consider multigraphs.
- We prove by induction on the number of edges.
  - **Base Step:** two edges
  - **Inductive Step:** A simple path of maximal length is a circuit, or otherwise the parity of the degrees at the start and end does not match. A simple circuit of maximal length is an Euler circuit, or otherwise we get an undirected multigraph whose vertices have even degree when we remove the circuit, from which we can construct an Euler circuit in a connected component that can be merged back to the original one.
Euler Circuits and Paths

Theorem
A connected undirected pseudograph with at least two vertices has an Euler circuit iff all of its vertices have even degree.

Proof Ideas (see [RD], Page 23)
- We only consider multigraphs.
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Proof Ideas (see [RD], Page 23)
- We only consider multigraphs.
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Euler Circuits and Paths

**Theorem**

A connected undirected pseudograph with at least two vertices has an **Euler circuit** iff all of its vertices have **even** degree.

**Proof Ideas (see [RD], Page 23)**

- We only consider multigraphs.
- We prove by induction on the number of edges.
- **Base Step:** two edges
Euler Circuits and Paths

**Theorem**

A connected undirected pseudograph with at least two vertices has an Euler circuit iff all of its vertices have even degree.

**Proof Ideas (see [RD], Page 23)**

- We only consider multigraphs.
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  - A simple path of maximal length is a circuit, or otherwise the parity of the degrees at the start and end does not match.
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A connected undirected pseudograph with at least two vertices has an Euler circuit iff all of its vertices have even degree.

Proof Ideas (see [RD], Page 23)

- We only consider multigraphs.
- We prove by induction on the number of edges.
- **Base Step:** two edges
- **Inductive Step:**
  - A simple path of maximal length is a circuit, or otherwise the parity of the degrees at the start and end does not match.
  - A simple circuit of maximal length is an Euler circuit, or otherwise we get an undirected multigraph whose vertices have even degree when we remove the circuit, from which we can construct an Euler circuit in a connected component that can be merged back to the original one.
Euler Circuits and Paths

Theorem

A connected undirected pseudograph has an Euler path iff there are at most two vertices with odd degree.
Euler Circuits and Paths

Theorem

A connected undirected pseudograph has an **Euler path** iff there are at most two vertices with **odd** degree.

Proof Idea (from [R], Page 697)

- **Case 1:** The graph does not have vertices of odd degree.
- **Case 2:** The graph have two distinct vertices of odd degree. Then we can add an edge between the two vertices so that all the vertices of the new graph have even degree and hence the new graph has an Euler circuit.
Euler Circuits and Paths

Examples

![Diagram](image)

- Left diagram: A graph with nodes a, b, c, d, e, showing a possible Euler circuit.
- Right diagram: A square graph with nodes a, b, c, d, e, illustrating another Euler circuit.

Connectivity, Paths and Circuits
Hamilton Paths and Circuits

[R], Page 698 – Page 703
Hamilton’s “A Voyage Round the World” Puzzle.
Hamilton’s “A Voyage Round the World” Puzzle.
Hamilton’s “A Voyage Round the World” Puzzle.

source: from “Discrete Mathematics” by Kenneth H. Rosen
Hamilton Paths and Circuits

Definition

- \( G = (V, E) \): a simple graph

Then we have:

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Hamilton Paths and Circuits

**Definition**

- $G = (V, E)$: a simple graph

Then we have:

- A Hamilton path is a simple path $v_0, v_1, \ldots, v_n$ of distinct vertices in $G$ such that $V = \{v_0, v_1, \ldots, v_n\}$;
- A Hamilton circuit is a simple circuit $v_0, v_1, \ldots, v_n, v_0$ ($n > 0$) such that $V = \{v_0, v_1, \ldots, v_n\}$ and $v_0, v_1, \ldots, v_n$ is a Hamilton path.
Hamilton Paths and Circuits

**Definition**

- \( G = (V, E) \): a simple graph

Then we have:

- a **Hamilton path** is a simple path \( v_0, v_1, \ldots, v_n \) of distinct vertices in \( G \) such that \( V = \{v_0, v_1, \ldots, v_n\} \);

- a **Hamilton circuit** is a simple circuit \( v_0, v_1, \ldots, v_n, v_0 (n > 0) \) such that \( V = \{v_0, v_1, \ldots, v_n\} \) and \( v_0, v_1, \ldots, v_n \) is a Hamilton path.
Hamilton Paths and Circuits

**Definition**

- \( G = (V, E) \): a simple graph

Then we have:

- a **Hamilton path** is a simple path \( v_0, v_1, \ldots, v_n \) of distinct vertices in \( G \) such that \( V = \{v_0, v_1, \ldots, v_n\} \);

- a **Hamilton circuit** is a simple circuit \( v_0, v_1, \ldots, v_n, v_0 \) (\( n > 0 \)) such that \( V = \{v_0, v_1, \ldots, v_n\} \) and \( v_0, v_1, \ldots, v_n \) is a Hamilton path.

**Examples**

![Graphs G1, G2, G3](image-url)
A More Sophisticated Example
Application: Gray Codes

- Hamilton circuits in $Q_n$

A More Sophisticated Example

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(a) [Diagram showing a Hamilton circuit]

(b) [Diagram showing another Hamilton circuit]
Problem

When does a simple graph have a Hamilton circuit?
Problem

When does a simple graph have a Hamilton circuit?

Bad News

- an NP-complete problem
- believably no efficient solution
Problem
When does a simple graph have a Hamilton circuit?

Bad News
- an NP-complete problem
- believably no efficient solution

Sufficient Conditions
- A dense graph that have many edges incident to every vertex tends to have a Hamilton circuit.
Dirac’s Theorem

- \( G = (V, E) \): a simple graph with at least three vertices

If it holds that \( \forall v \in V (\deg(v) \geq \frac{|V|}{2}) \), then \( G \) has a Hamilton circuit.
### Dirac’s Theorem

- \( G = (V, E) \): a simple graph with at least three vertices

If it holds that \( \forall v \in V \ (\deg(v) \geq \frac{|V|}{2}) \), then \( G \) has a Hamilton circuit.

### Ore’s Theorem

- \( G = (V, E) \): a simple graph with at least three vertices

If it holds that

\[
\forall u, v \in V \ [(u \neq v \land \{u, v\} \not\in E) \rightarrow (\deg(u) + \deg(v) \geq |V|)]
\]

then \( G \) has a Hamilton circuit.
Hamilton Paths and Circuits

Dirac’s Theorem

- $G = (V, E)$: a simple graph with at least three vertices

If it holds that $\forall v \in V \ (\text{deg}(v) \geq \frac{|V|}{2})$, then $G$ has a Hamilton circuit.

Ore’s Theorem

- $G = (V, E)$: a simple graph with at least three vertices

If it holds that

$$\forall u, v \in V \ [ (u \neq v \land \{u, v\} \not\in E) \rightarrow (\text{deg}(u) + \text{deg}(v) \geq |V|) ]$$

then $G$ has a Hamilton circuit.

Remark

Ore’s Theorem is not necessary since we have the cycles $C_n$’s ($n \geq 5$).
Hamilton Paths and Circuits

Ore’s Theorem

- \( G = (V, E) \): a simple graph with \( n \geq 3 \) vertices

If \( \forall u, v \in V \ [(u \neq v \land \{u, v\} \not\in E) \rightarrow (\deg(u) + \deg(v) \geq n)] \), then \( G \) has a Hamilton circuit.

Proof Idea (by Contradiction)

Suppose that \( G \) does not have a Hamilton circuit. Construct a graph \( H \) such that (i) \( G \subseteq H \), (ii) \( H \) does not have a Hamilton circuit, (iii) \( H + e \) has a Hamilton circuit for every edge \( e \).

\( H \) has a Hamilton path \( v_1, \ldots, v_n \) where \( \deg(v_1) + \deg(v_n) \geq n \).

For the Hamilton path \( v_1, \ldots, v_n \), the case \(|V| = 3\) is straightforward. For \( n \geq 4 \), by Pigeon-Hole principle, there is \( 2 \leq i \leq n - 2 \) such that \( \{v_i, v_n\}, \{v_i+1, v_1\} \in E \).

Then \( v_1, \ldots, v_i, v_n, v_n-1, \ldots, v_i+1, v_1 \) is however a Hamilton circuit.
Ore’s Theorem

- \( G = (V, E) \): a simple graph with \( n \geq 3 \) vertices

If \( \forall u, v \in V \ [(u \neq v \land \{u, v\} \notin E) \rightarrow (\deg(u) + \deg(v) \geq n)] \), then \( G \) has a Hamilton circuit.

Proof Idea (by Contradiction)

- Suppose that \( G \) does not have a Hamilton circuit.
Hamilton Paths and Circuits

Ore’s Theorem

- \( G = (V, E) \): a simple graph with \( n \geq 3 \) vertices

If \( \forall u, v \in V \ [(u \neq v \land \{u, v\} \notin E) \rightarrow (\text{deg}(u) + \text{deg}(v) \geq n)] \), then \( G \) has a Hamilton circuit.

Proof Idea (by Contradiction)

- Suppose that \( G \) does not have a Hamilton circuit.
- Construct a graph \( H \) such that (i) \( G \subseteq H \), (ii) \( H \) does not have a Hamilton circuit, (iii) \( H + e \) has a Hamilton circuit for every edge \( e \).
Hamilton Paths and Circuits

Ore’s Theorem

- \( G = (V, E) \): a simple graph with \( n \geq 3 \) vertices

If \( \forall u, v \in V \ [(u \neq v \land \{u, v\} \not\in E) \rightarrow (\text{deg}(u) + \text{deg}(v) \geq n)] \), then \( G \) has a Hamilton circuit.

Proof Idea (by Contradiction)

- Suppose that \( G \) does not have a Hamilton circuit.
- Construct a graph \( H \) such that (i) \( G \subseteq H \), (ii) \( H \) does not have a Hamilton circuit, (iii) \( H + e \) has a Hamilton circuit for every edge \( e \).
- \( H \) has a Hamilton path \( v_1, \ldots, v_n \) where \( \text{deg}(v_1) + \text{deg}(v_n) \geq n \).
Ore’s Theorem

- \( G = (V, E) \): a simple graph with \( n \geq 3 \) vertices

If \( \forall u, v \in V \ [(u \neq v \wedge \{u, v\} \not\in E) \rightarrow (\deg(u) + \deg(v) \geq n)] \), then \( G \) has a Hamilton circuit.

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- For the Hamilton path \( v_1, \ldots, v_n \), the case \( |V| = 3 \) is straightforward.
**Ore’s Theorem**

- \( G = (V, E) \): a simple graph with \( n \geq 3 \) vertices

If \( \forall u, v \in V \ [(u \neq v \land \{u, v\} \notin E) \rightarrow (\deg(u) + \deg(v) \geq n)] \), then \( G \) has a Hamilton circuit.

**Proof Idea (by Contradiction)**

- Suppose that \( G \) does not have a Hamilton circuit.
- Construct a graph \( H \) such that (i) \( G \subseteq H \), (ii) \( H \) does not have a Hamilton circuit, (iii) \( H + e \) has a Hamilton circuit for every edge \( e \).
- \( H \) has a Hamilton path \( v_1, \ldots, v_n \) where \( \deg(v_1) + \deg(v_n) \geq n \).
- For the Hamilton path \( v_1, \ldots, v_n \), the case \( |V| = 3 \) is straightforward.
- For \( n \geq 4 \), by Pigeon-Hole principle, there is \( 2 \leq i \leq n - 2 \) such that \( \{v_i, v_n\}, \{v_{i+1}, v_1\} \in E \).
Hamilton Paths and Circuits

Ore’s Theorem

- \( G = (V, E) \): a simple graph with \( n \geq 3 \) vertices

If \( \forall u, v \in V \ [(u \neq v \land \{u, v\} \notin E) \rightarrow (\deg(u) + \deg(v) \geq n)] \), then \( G \) has a Hamilton circuit.

Proof Idea (by Contradiction)

- Suppose that \( G \) does not have a Hamilton circuit.
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- \( H \) has a Hamilton path \( v_1, \ldots, v_n \) where \( \deg(v_1) + \deg(v_n) \geq n \).
- For the Hamilton path \( v_1, \ldots, v_n \), the case \( |V| = 3 \) is straightforward.
- For \( n \geq 4 \), by Pigeon-Hole principle, there is \( 2 \leq i \leq n - 2 \) such that \( \{v_i, v_n\}, \{v_{i+1}, v_1\} \in E \).
- Then \( v_1, \ldots, v_i, v_n, v_{n-1}, \ldots, v_{i+1}, v_1 \) is however a Hamilton circuit.
• paths, circuits and connectivity
• Euler paths and circuits
• Hamilton paths and circuits
main textbook: 

reference material: 
《图论与代数结构》，戴一奇，清华大学出版社

reference material: 
Reinhard Diestel, *Graph Theory*, 5th edition [RD] 
http://diestel-graph-theory.com/basic.html?
[R], Page 678 – 689, Page 693 – 703

(optional) 《图论与代数结构》，戴一奇，清华大学出版社
Homeworks

Connectivity

- [R], Page 689, Exercise 2(a)(b)(d), 6
- [R], Page 690, 14(a)(b)(c)
- [R], Page 692, Exercise 50(a)(c)(d), Exercise 52(b), Exercise 63

Note for Exercise 52(b): Please prove without using the inequalities \( \kappa(G) \leq \lambda(G) \leq \min_{v \in V} \deg(v) \). You can proceed by induction on the number of vertices.

Hint for Exercise 63: Fix any vertex \( v \) and consider partition the vertices into those which can be connected from \( v \) through a path of odd length and those which can be connected from \( v \) through a path of even length.
Homeworks

**Euler Paths and Circuits**
- [R], Page 704, Exercise 6, 8

**Hamilton Paths and Circuits**
- [R], Page 705, Exercise 36, 40
- [R], Page 706, Exercise 49, 55
Homework Submission

- **submission time:** the start of the class on Nov. 19th
- **teaching assistant:**
  - Peixin Wang: peter007008@qq.com
  - Jinyi Wang: jinyi.wang@sjtu.edu.cn
  - Luhua Jin: 1097795310@qq.com
- **submission:**
  - written version (preferred): submit on the desk
  - electronic version: word or pdf version, send email with title

“离散数学+姓名+学号+第十周周二”

to the teaching assistants:
- Students from F1903001, F1903003 and F1903004, send to Luhua Jin.
- Students from F1903801 and F1903802, send to Jinyi Wang.
- All other students please send to Peixin Wang.