Planar Graphs and Graph Coloring

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- paths, circuits and connectivity
- Euler paths and circuits
- Hamilton paths and circuits
Today’s Topic

- planar graphs
- graph coloring
Planar Graphs (平面图)

main textbook, Page 718 – 725
Can we embed the following electronic circuit onto a board?

source: from internet
Definition (for rigorous definition see [RD])

\[ G = (V, E) \]: an undirected multigraph

The graph \( G \) is planar if it can be drawn in the plane without any edge crossing, where an edge crossing is an intersection point of two edges other than their common endpoints.
Planar Graphs

**Definition (for rigorous definition see [RD])**

- \( G = (V, E) \): an undirected multigraph

The graph \( G \) is **planar** if it can be drawn in the plane without any edge crossing, where an edge crossing is an intersection point of two edges other than their common endpoints.

**Positive Examples**
- the complete graph \( K_4 \)
- the three dimensional cube \( Q_3 \)

**Negative Examples**
- the bipartite complete graph \( K_{3,3} \)
- the complete graph \( K_5 \)
Examples

Planarity of $K_4$ and $Q_3$
Examples

**Planarity of $K_4$ and $Q_3$**

**Non-planarity of $K_{3,3}$**
The Central Problem

How can one judge whether a simple graph is planar?
Planar Graphs

Regions

A planar graph splits the plane into regions through its vertices and edges.
Planar Graphs

Regions

A planar graph splits the plane into regions through its vertices and edges.

Example

![Diagram of a planar graph with regions labeled R1, R2, R3, R4, R5, and R6.]}
Euler’s Formula

Let $G$ be a connected simple planar graph with $n$ vertices, $m$ edges and $\ell$ regions. Then we have that

$$\ell = m - n + 2.$$
Euler’s Formula

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Proof Ideas

- Construct connected simple graphs $G_1, \ldots, G_m$ such that (i) $G_1$ consists of only one vertex and $G_m = G$, (ii) $G_{k+1}$ is obtained from $G_k$ by adding one edge incident with some vertex in $G_k$.
- Show that the formula is preserved at each step from $G_k$ to $G_{k+1}$. 
Euler’s Formula

Let \( G \) be a connected simple planar graph with \( n \) vertices, \( m \) edges and \( \ell \) regions. Then we have that

\[
\ell = m - n + 2.
\]

Proof Ideas

- Construct connected simple graphs \( G_1, \ldots, G_m \) such that (i) \( G_1 \) consists of only one vertex and \( G_m = G \), (ii) \( G_{k+1} \) is obtained from \( G_k \) by adding one edge incident with some vertex in \( G_k \).
- Show that the formula is preserved at each step from \( G_k \) to \( G_{k+1} \).

Example

![Diagram of a planar graph showing regions labeled R1 to R6.](image)
Corollary

For any connected simple planar graph with $n$ vertices and $m$ edges, if $n \geq 3$, then $m \leq 3 \cdot n - 6$. 

Degree of a Region

The degree of a region $R$, denoted by $\deg(R)$, is the number of edges lying on the boundary of $R$. An edge whose both sides are the same region contributes two to the degree of the region.

Proof Ideas

Each region has a degree at least three. 

$$2 \cdot m = \sum \deg(R) \geq 3 \cdot \ell.$$ 

Hence 

$$\frac{2}{3} \cdot m \geq \ell.$$ 

$$m \geq \frac{3}{2} (m - n + 2).$$ 

Hence 

$$m \leq 3 \cdot n - 6.$$
Corollary

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Proof Ideas

- Each region has a degree at least three.
- $2 \cdot m = \sum_{R} \deg(R) \geq 3 \cdot \ell$. Hence $\frac{2}{3} \cdot m \geq \ell$.
- $\frac{2}{3} \cdot m \geq m - n + 2$. Hence $m \leq 3 \cdot n - 6$. 
Corollary

For any connected simple planar graph with \( n \) vertices and \( m \) edges, if \( n \geq 3 \), then \( m \leq 3 \cdot n - 6 \).

Corollary

- \( K_5 \) is not planar.
- Any connected simple planar graph contains a vertex of degree less than 6.
Corollary

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Corollary

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- Any connected simple planar graph contains a vertex of degree less than 6.

Question

What about \( K_{3,3} \) ?
Corollary

For any connected simple planar graph with $n$ vertices and $m$ edges, if $n \geq 3$ and any simple circuit in the graph has at least 4 edges, then $m \leq 2 \cdot n - 4$. 
Corollary
For any connected simple planar graph with $n$ vertices and $m$ edges, if $n \geq 3$ and any simple circuit in the graph has at least 4 edges, then $m \leq 2 \cdot n - 4$.

Corollary
The complete bipartite graph $K_{3,3}$ is not planar.
Corollary

For any connected simple planar graph with \( n \) vertices and \( m \) edges, if \( n \geq 3 \) and any simple circuit in the graph has at least 4 edges, then \( m \leq 2 \cdot n - 4 \).

Corollary

The complete bipartite graph \( K_{3,3} \) is not planar.

Kuratowski’s Theorem

\( K_5 \) and \( K_{3,3} \) together characterizes the class of all planar simple graphs.
The Operation of Elementary Subdivision

- the original edge: \( \{u, v\} \)
- new edges: \( \{u, w\} \) and \( \{w, v\} \)
The Operation of Elementary Subdivision

- the original edge: \( \{u, v\} \)
- new edges: \( \{u, w\} \) and \( \{w, v\} \)

Homeomorphism （同胚）

- \( G_i = (V_i, E_i) \) (\( i = 1, 2 \)): two simple graphs

We say that \( G_1 \) and \( G_2 \) are homeomorphic if they can be obtained from the same graph through elementary subdivisions.
Kuratowski’s Theorem

A simple graph is nonplanar iff it has a subgraph homeomorphic to either $K_{3,3}$ or $K_5$. 
Kuratowski’s Theorem

A simple graph is nonplanar iff it has a subgraph homeomorphic to either $K_{3,3}$ or $K_5$.

Proof Ideas (Sketch) (details in the textbook [RD])

- Every 3-connected graph not having a subgraph homeomorphic to either $K_{3,3}$ or $K_5$ is planar.
- Every graph which is edge-maximal not to have a subgraph homeomorphic to either $K_{3,3}$ or $K_5$ is 3-connected.
Examples

- the cube $Q_3$ with an extra diagonal edge
- the Petersen Graph
Planar Graphs

Examples

- the cube $Q_3$ with an extra diagonal edge
- the Petersen Graph

The Petersen Graph
Planar Graphs

**Algorithms**

- Highly efficient algorithms
- Linear time (in the number of vertices)
Graph Coloring

main textbook, Page 727 – 732
Graph Coloring

Dual Graphs （对偶图）

- $G$: a planar undirected multigraph

The dual graph of $G$ is an undirected multigraph obtained by:
- first assigning to each region $r_i$ a vertex $\bar{r}_i$;
- then for each edge $e$ in $G$ that separates regions $r_i$, $r_j$, there is an edge $\bar{e}$ that connects $\bar{r}_i$, $\bar{r}_j$;
- when $i = j$, $\bar{e}$ is a self-loop.
Graph Coloring

Dual Graphs （对偶图）

- \( G \): a planar undirected multigraph

The dual graph of \( G \) is an undirected multigraph obtained by:

- first assigning to each region \( r_i \) a vertex \( \overline{r_i} \);
- then for each edge \( e \) in \( G \) that separates regions \( r_i, r_j \), there is an edge \( \overline{e} \) that connects \( \overline{r_i}, \overline{r_j} \);
- when \( i = j \), \( \overline{e} \) is a self-loop.

Problem

How many colors do we need to color a map (as a dual graph) so that adjacent countries have different colors?
A coloring of a simple graph is a function that assigns a color to each vertex in the graph so that adjacent vertices have different colors.
Graph Coloring

A coloring of a simple graph is a function that assigns a color to each vertex in the graph so that adjacent vertices have different colors.

Chromatic Number

The chromatic number of a simple graph $G$, denoted by $\chi(G)$, is the least number of distinct colors required for a coloring of $G$. 
Graph Coloring

A coloring of a simple graph is a function that assigns a color to each vertex in the graph so that adjacent vertices have different colors.

Chromatic Number

The chromatic number of a simple graph $G$, denoted by $\chi(G)$, is the least number of distinct colors required for a coloring of $G$.

Example

![Graph Coloring Diagram]
The Four-Color Theorem (四色定理)

The chromatic number of any planar graph is no greater than four.
The Four-Color Theorem （四色定理）

The chromatic number of any planar graph is no greater than four.

- **mechanical proof** by computer
Summary

- planar graphs
- graph coloring
Textbooks

- **main textbook:** Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th edition [R]
- **reference material:** 《图论与代数结构》，戴一奇等，清华大学出版社
- **reference material:** Reinhard Diestel, *Graph Theory*, 5th edition [RD]
  
  http://diestel-graph-theory.com/basic.html?
- [R], Page 718 – 732
- (optional) 《图论与代数结构》，戴一奇等，清华大学出版社
Judge whether the following simple graphs are planar or not:

- [R], Page 725, Exercise 7, 8, 9
- [R], Page 726, Exercise 23, 25

If the graph is planar, present a planar drawing of the graph. Otherwise, use Kuratowski’s Theorem to prove that it is not planar.
Homework Submission

- **submission time:** the start of the class on Nov. 26th
- **teaching assistant:**
  - Peixin Wang: peter007008@qq.com
  - Jinyi Wang: jinyi.wang@sjtu.edu.cn
  - Luhua Jin: 1097795310@qq.com
- **submission:**
  - written version (preferred): submit on the desk
  - electronic version: word or pdf version, send email with title

“离散数学+姓名+学号+第十一周周二”

to the teaching assistants:
  - Students from F1903001, F1903003 and F1903004, send to Luhua Jin.
  - Students from F1903801 and F1903802, send to Jinyi Wang.
  - All other students please send to Peixin Wang.