Finishing Graph Theory

- Huffman coding
- proof of correctness
Today’s Topic

Some Combinatorics

- inclusion-exclusion principle
- (optional) solving linear recurrences
Inclusion-Exclusion Principle (容斥原理)

main textbook, Page 552 – 564
Inclusion-Exclusion Principle

Motivation

- $A, B, C$: finite sets

Then we have:

\[
\begin{align*}
|A \cup B| &= |A| + |B| - |A \cap B|, \\
|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.
\end{align*}
\]
Inclusion-Exclusion Principle

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- $|A \cup B| = |A| + |B| - |A \cap B|$;
- $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$.
An Example

- How many natural numbers between 0 and 100 (inclusive) that are divisible either by 2 or 3 or 5?
Inclusion-Exclusion Principle

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- \( A = \{ n \in \mathbb{N} \mid 2 \mid n \} \)
- \( B = \{ n \in \mathbb{N} \mid 3 \mid n \} \)
- \( C = \{ n \in \mathbb{N} \mid 5 \mid n \} \)

\[ |A \cup B \cup C| = 51 + 34 + 21 - 17 - 7 - 11 + 4. \]
\[ |A \cup B \cup C| = 75. \]
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Inclusion-Exclusion Principle

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- \(| A \cup B \cup C | = | A | + | B | + | C | - | A \cap B | - | B \cap C | - | C \cap A | + | A \cap B \cap C |\).
- \(| A \cup B \cup C | = 51 + 34 + 21 - 17 - 7 - 11 + 4.\)
- \(| A \cup B \cup C | = 75\).
**Theorem Statement**

- \( A_1, \ldots, A_n \): finite sets

Then we have that

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|\bigcup_{i=1}^{n} A_i| = \sum_{\ell=1}^{n} \sum_{1 \leq k_1 < \cdots < k_\ell \leq n} (-1)^{\ell+1} \cdot |\bigcap_{j=1}^{\ell} A_{k_j}|.
\]
Inclusion-Exclusion Principle

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**Proof**

We show that every element in $\bigcup_{i=1}^n A_i$ is counted in total once in the right hand side of the equality.
Inclusion-Exclusion Principle

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We show that every element in $\bigcup_{i=1}^{n} A_i$ is counted in total once in the right hand side of the equality.

- Suppose that an element $u$ appears in $A_{b_1}, \ldots, A_{b_m}$. 

Inclusion-Exclusion Principle

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We show that every element in \( \bigcup_{i=1}^{n} A_i \) is counted in total once in the right hand side of the equality.

- Suppose that an element \( u \) appears in \( A_{b_1}, \ldots, A_{b_m} \).
- For each \( \ell \leq m \), the summation \( \sum_{1 \leq k_1 < \ldots < k_\ell \leq n} (-1)^{\ell+1} \cdot | \bigcap_{j=1}^{\ell} A_{k_j} | \) counts the element \( u \) up to \((-1)^{\ell+1} \cdot C_{m}^{\ell} \).
Inclusion-Exclusion Principle

**Theorem Statement**

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$$|\bigcup_{i=1}^n A_i| = \sum_{\ell=1}^n \sum_{1\leq k_1<\ldots<k_\ell\leq n} (-1)^{\ell+1} \cdot |\bigcap_{j=1}^\ell A_{k_j}|.$$  

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We show that every element in $\bigcup_{i=1}^n A_i$ is counted in total once in the right hand side of the equality.

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- For each $\ell \leq m$, the summation $\sum_{1\leq k_1<\ldots<k_\ell\leq n} (-1)^{\ell+1} \cdot |\bigcap_{j=1}^\ell A_{k_j}|$ counts the element $u$ up to $(-1)^{\ell+1} \cdot C^\ell_m$.
- In total, $u$ is counted up to $\sum_{\ell=1}^m (-1)^{\ell+1} \cdot C^\ell_m = 1$. 

Inclusion-Exclusion Principle

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| \bigcup_{i=1}^{n} A_i | = \sum_{\ell=1}^{n} \sum_{1 \leq k_1 < \cdots < k_\ell \leq n} (-1)^{\ell+1} \cdot | \bigcap_{j=1}^{\ell} A_{k_j} |
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**Derangement**

- **problem:** how many bijections \( f : \{1, \ldots, n\} \to \{1, \ldots, n\} \) are there such that \( \forall n. f(n) \neq n \)?
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- \( |\bigcup_{i=1}^{n} A_i| = \sum_{\ell=1}^{n} \sum_{1 \leq k_1 < \cdots < k_\ell \leq n} (-1)^{\ell+1} \cdot (n - \ell)! \)
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- \( \left| \bigcup_{i=1}^{n} A_i \right| = \sum_{\ell=1}^{n} C_{n}^{\ell} \cdot (-1)^{\ell+1} \cdot (n - \ell)! \)
Theorem Statement

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Derangement

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**\( A_k \):** \( \{ f \mid f \text{ bijection, } f(k) = k \} \) \((1 \leq k \leq n)\)

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\[ |\bigcup_{i=1}^{n} A_i| = \sum_{\ell=1}^{n} (-1)^{\ell+1} \cdot \frac{n!}{\ell!} = n! \cdot (\sum_{\ell=1}^{n} (-1)^{\ell+1} \cdot \frac{1}{\ell!}) \]
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The answer is \( n! \cdot \sum_{\ell=0}^{n} (-1)^{\ell} \cdot \frac{1}{\ell!} \).
### Inclusion-Exclusion Principle

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#### Onto Functions

- **Problem:** how many onto functions \( f : \{1, \ldots, m\} \rightarrow \{1, \ldots, n\} \) are there?
Inclusion-Exclusion Principle

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**Onto Functions**

- **Problem**: How many onto functions \( f : \{1, \ldots, m\} \to \{1, \ldots, n\} \) are there?
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### Inclusion-Exclusion Principle

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Onto Functions

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**Euler’s Totient Function**

- **Problem:** How many integers from \(\{0, 1, \ldots, m - 1\}\) (\(m \geq 2\)) are relatively prime to \(m = p_1^{\alpha_1} \cdots p_n^{\alpha_n}\)? (\(p_i \geq 2\) are distinct prime numbers)
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\[ | \bigcup_{i=1}^{n} A_i | = m \cdot \left(1 - \frac{1}{p_1}\right) \cdot \ldots \cdot \left(1 - \frac{1}{p_n}\right) \]
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\[ |\bigcup_{i=1}^{n} A_i| = \sum_{\ell=1}^{n} \sum_{1 \leq k_1 < \ldots < k_\ell \leq n} (-1)^{\ell+1} \cdot |\bigcap_{j=1}^{\ell} A_{k_j}| \]

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- \(|\bigcup_{i=1}^{n} A_i| = m - m \cdot \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_n}\right)\)
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**Euler’s Totient Function**

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- The answer is \( m \cdot \left(1 - \frac{1}{p_1}\right) \cdot \ldots \cdot \left(1 - \frac{1}{p_n}\right) \).
Solving Linear Recurrences

main textbook, Page 514 – 524
Solving Linear Recurrences

Problem
Given a linear recurrence relation

\[ a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \cdots + c_k \cdot a_{n-k} \]

where \( c_1, \ldots, c_k \) are real constants (\( c_k \neq 0 \)), how can we solve it exactly given the initial values for \( a_0, \ldots, a_{k-1} \)?
Solving Linear Recurrences

**Theorem**

- \( a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} \): a linear recurrence relation
- \( a_0, a_1 \): known constants
- \( r_1, r_2 \): two distinct roots of the equation \( r^2 - c_1 \cdot r - c_2 = 0 \)

Then \( a_n = \alpha_1 \cdot r_1^n + \alpha_2 \cdot r_2^n \) for \( n \geq 0 \), where \( \alpha_1, \alpha_2 \) are constants uniquely determined by \( a_0, a_1 \).
Fibonacci Numbers

- \( a_n = a_{n-1} + a_{n-2} \)
- \( a_0 = 0, \ a_1 = 1 \)

We solve as follows:
Solving Linear Recurrences

Fibonacci Numbers

- \( a_n = a_{n-1} + a_{n-2} \)
- \( a_0 = 0, a_1 = 1 \)

We solve as follows:
- \( r_1 = \frac{1-\sqrt{5}}{2}, \quad r_2 = \frac{1+\sqrt{5}}{2} \) from the equation \( r^2 - 1 \cdot r - 1 = 0 \)
Fibonacci Numbers

- \( a_n = a_{n-1} + a_{n-2} \)
- \( a_0 = 0, \ a_1 = 1 \)

We solve as follows:

- \( r_1 = \frac{1-\sqrt{5}}{2}, \ r_2 = \frac{1+\sqrt{5}}{2} \) from the equation \( r^2 - 1 \cdot r - 1 = 0 \)
- \( a_n = \alpha_1 \cdot r_1^n + \alpha_2 \cdot r_2^n \)
Fibonacci Numbers

- \( a_n = a_{n-1} + a_{n-2} \)
- \( a_0 = 0, a_1 = 1 \)

We solve as follows:

- \( r_1 = \frac{1-\sqrt{5}}{2}, \ r_2 = \frac{1+\sqrt{5}}{2} \) from the equation \( r^2 - 1 \cdot r - 1 = 0 \)
- \( a_n = \alpha_1 \cdot r_1^n + \alpha_2 \cdot r_2^n \)
- \( \alpha_1 + \alpha_2 = 0, \ \alpha_1 \cdot r_1 + \alpha_2 \cdot r_2 = 1 \)
Fibonacci Numbers

- \( a_n = a_{n-1} + a_{n-2} \)
- \( a_0 = 0, a_1 = 1 \)

We solve as follows:

- \( r_1 = \frac{1-\sqrt{5}}{2}, \quad r_2 = \frac{1+\sqrt{5}}{2} \) from the equation \( r^2 - 1 \cdot r - 1 = 0 \)
- \( a_n = \alpha_1 \cdot r_1^n + \alpha_2 \cdot r_2^n \)
- \( \alpha_1 + \alpha_2 = 0, \quad \alpha_1 \cdot r_1 + \alpha_2 \cdot r_2 = 1 \)
- \( \alpha_1 = -\frac{1}{\sqrt{5}}, \quad \alpha_2 = \frac{1}{\sqrt{5}} \)
Fibonacci Numbers

- \( a_n = a_{n-1} + a_{n-2} \)
- \( a_0 = 0, \ a_1 = 1 \)

We solve as follows:

- \( r_1 = \frac{1-\sqrt{5}}{2}, \ r_2 = \frac{1+\sqrt{5}}{2} \) from the equation \( r^2 - 1 \cdot r - 1 = 0 \)
- \( a_n = \alpha_1 \cdot r_1^n + \alpha_2 \cdot r_2^n \)
- \( \alpha_1 + \alpha_2 = 0, \ \alpha_1 \cdot r_1 + \alpha_2 \cdot r_2 = 1 \)
- \( \alpha_1 = -\frac{1}{\sqrt{5}}, \ \alpha_2 = \frac{1}{\sqrt{5}} \)
- \( a_n = \left( -\frac{1}{\sqrt{5}} \right) \cdot r_1^n + \frac{1}{\sqrt{5}} \cdot r_2^n \)
Another Example

- $a_n = -a_{n-2}$
- $a_0 = 0, a_1 = 1$

We solve as follows:

- $r_1 = i, r_2 = -i$ from the equation $r^2 + 1 = 0$
- $a_n = \alpha_1 \cdot r_1^n + \alpha_2 \cdot r_2^n$
- $\alpha_1 + \alpha_2 = 0, \alpha_1 \cdot r_1 + \alpha_2 \cdot r_2 = 1$
- $\alpha_1 = -\frac{i}{2}, \alpha_2 = \frac{i}{2}$
- $a_n = \left(-\frac{i}{2}\right) \cdot i^n + \frac{i}{2} \cdot (-i)^n$
Solving Linear Recurrences

**Theorem**

- \( a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} \ (c_2 \neq 0) \)
- \( a_0, a_1 \): known constants
- \( r \): the unique root of the equation \( r^2 - c_1 \cdot r - c_2 = 0 \)

Then \( a_n = \alpha_1 \cdot r^n + \alpha_2 \cdot n \cdot r^n \) for \( n \geq 0 \), where \( \alpha_1, \alpha_2 \) are constants uniquely determined by \( a_0, a_1 \).
An Example

\[ a_n = 6 \cdot a_{n-1} - 9 \cdot a_{n-2} \]
\[ a_0 = 0, \quad a_1 = 1 \]

We solve as follows:

\[ r = 3 \text{ from the equation } r^2 - 6 \cdot r + 9 = 0 \]
\[ a_n = \alpha_1 \cdot r^n + \alpha_2 \cdot n \cdot r^n \]
\[ \alpha_1 = 0, \quad \alpha_1 \cdot r + \alpha_2 \cdot r = 1 \]
\[ \alpha_1 = 0, \quad \alpha_2 = \frac{1}{3} \]
\[ a_n = \frac{1}{3} \cdot n \cdot 3^n \]
The General Theorems

See Page 518–519, [R].
Summary

- inclusion-exclusion principle
- solving linear recurrences
Textbooks

- **main textbook:**
[R], Page 552 – 564, (optional) Page 514 – 524

(optional) The Pigeonhole Principle:
  [R], Page 399 – 405
  Dirichlet’s Approximation Theorem

(optional) Generating Functions:
  [R], Page 537 – 548
  Catalan numbers
  https://en.wikipedia.org/wiki/Catalan_number
Homeworks

- [R], Page 557, Exercise 8
- [R], Page 558, Exercise 22
- [R], Page 565, Exercise 13, 16, 17
Homework Submission

- **submission time**: the start of the class on Dec. 10th
- **teaching assistant**:
  - Peixin Wang: peter007008@qq.com
  - Jinyi Wang: jinyi.wang@sjtu.edu.cn
  - Luhua Jin: 1097795310@qq.com
- **submission**:
  - written version (preferred): submit on the desk
  - electronic version: word or pdf version, send email with title

“离散数学+姓名+学号+第十三周周五”

to the teaching assistants:
- Students from F1903001, F1903003 and F1903004, send to Luhua Jin.
- Students from F1903801 and F1903802, send to Jinyi Wang.
- All other students please send to Peixin Wang.