CS257 Linear and Convex Optimization
Homework 10

Due: November 30, 2020

November 23, 2020

1. Consider the equality constrained least squares problem

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \|\boldsymbol{A}\boldsymbol{x}-\boldsymbol{b}\|^2 \\ \text{s.t.} \quad & \boldsymbol{G}\boldsymbol{x}=\boldsymbol{h} \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ with $\operatorname{rank} \boldsymbol{A} = n$, $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{G} \in \mathbb{R}^{k\times n}$ with $\operatorname{rank} \boldsymbol{G} = k$.

(a). Find KKT system of this problem

(b). Find a closed form solution for the optimal solution $\boldsymbol{x}^*$ and the corresponding Lagrange multiplier $\boldsymbol{\lambda}^*$.

(c). Find the KKT system and $\boldsymbol{x}^*$ and $\boldsymbol{\lambda}^*$ for

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ 1 & 2 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 5 \\ 10 \\ -5 \end{bmatrix}, \quad \boldsymbol{G} = (1,1), \quad \boldsymbol{h} = 1$$

(d). Find the optimal solution $\boldsymbol{x}^*$ for the $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{G}, \boldsymbol{h}$ in (c) by reduction to an unconstrained problem.

2. We considered the following convex optimization problem

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x}=\boldsymbol{b} \end{aligned} \tag{1}$$

Suppose $\boldsymbol{Q} \succeq \boldsymbol{O}$. The problem

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & h(\boldsymbol{x}) \triangleq f(\boldsymbol{x}) + (\boldsymbol{A}\boldsymbol{x}-\boldsymbol{b})^T\boldsymbol{Q}(\boldsymbol{A}\boldsymbol{x}-\boldsymbol{b}) \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x}=\boldsymbol{b} \end{aligned} \tag{2}$$

is equivalent to the problem in (1).

(a). Suppose we want to use Newton's method to solve (2). Find the KKT system used to solve for the Newton direction at a feasible $\boldsymbol{x}_0$.

(b). If we use Newton's method to solve both (1) and (2), are the Newton directions the same at the same feasible point $\boldsymbol{x}$? Are the corresponding Lagrange multipliers (the $\boldsymbol{\lambda}$ in the KKT system) the same?

3. Let $\boldsymbol{x} \in \mathbb{R}^3$. Consider

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & f(\boldsymbol{x}) = e^{x_1} + e^{2x_2} + e^{2x_3} \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 1 \end{aligned} \tag{3}$$

(a). Solve problem (3) by the Lagrange multiplier method. Show the optimal solution $\boldsymbol{x}^*$, the Lagrange multiplier λ^* and the optimal value f^*.

(b). Find the closed-form expression for the Newton direction at a feasible $\boldsymbol{x}$ by solving the KKT system. Write a simple loop to run 5 iterations of Newton's method with the direction you find, initial point $\boldsymbol{x}_0 = (1, 0, 0)^T$, and constant step size $t = 1$. Show $\boldsymbol{x}$ for each iteration.

(c). Implement the algorithm on slide 20 of Lecture 12 in the `newton_eq` function of `newton.py`. Then use your implementation to solve (3) with initial points $\boldsymbol{x}_0 = (1, 0, 0)^T$ and $\boldsymbol{x}_0 = (2, -1, 0)^T$. Show the outputs.