CS257 Linear and Convex Optimization

Homework 13

Due: December 23, 2020

December 14, 2020

1. Consider the following problem

$$\begin{aligned} \min_{\boldsymbol{x} \in \mathbb{R}^2} \quad & f(\boldsymbol{x}) = x_1^2 + x_2^2 \\ \text{s.t.} \quad & (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1 \\ & (x_1 - 1)^2 + (x_2 + 1)^2 \leq 1 \end{aligned}$$

(a). Sketch the feasible set and the level sets of the objective function. Find the optimal point $\boldsymbol{x}^*$ and the optimal value f^*.

(b). Find the Lagrange dual function and the dual problem. Find the dual optimal value ϕ^*. Does strong duality hold?

(c). Does Slater's condition hold? What can you conclude about the necessity of Slater's condition for strong duality?

(d). Is the dual optimal value ϕ^* attained by any dual feasible point? Do there exist any Lagrange multipliers satisfying the KKT conditions at $\boldsymbol{x}^*$?

2. Consider the following minimization problem

$$\begin{aligned} \min_{\boldsymbol{x} \in \mathbb{R}^2} \quad & f(\boldsymbol{x}) = \begin{cases} x_1^3 + x_2^3, & \text{if } \boldsymbol{x} \geq \boldsymbol{0} \\ +\infty, & \text{otherwise} \end{cases} \\ \text{s.t.} \quad & x_1 + x_2 \geq 1 \end{aligned} \tag{P1}$$

Note the domain of f is $\operatorname{dom} f = \{\boldsymbol{x} \in \mathbb{R}^2 : \boldsymbol{x} \geq \boldsymbol{0}\}$ and the domain of the problem is $D = \operatorname{dom} f$.

(a). Since D is not the entire space, the dual function of this problem is defined by

$$\phi(\mu) = \inf_{\boldsymbol{x} \in D} \{f(\boldsymbol{x}) + \mu(1 - x_1 - x_2)\}$$

Find the explicit expression of $\phi(\mu)$.

(b). Find the dual optimal solution.

(c). What is the primal optimal value? Hint: Note f is convex on its domain.

(d). Note the primal problem (P1) is equivalent to

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{R}^2} \quad & f_1(\boldsymbol{x}) = x_1^3 + x_2^3 \\ \text{s.t.} \quad & x_1 + x_2 \geq 1 \\ & \boldsymbol{x} \geq \mathbf{0} \end{aligned} \tag{P2}$$

What's the dual function of this equivalent problem (P2)? Does strong duality hold for (P2)?

Remark. Equivalent primal problems can have very different dual problems. Not all dual problems are equally useful.