# CS257 Linear and Convex Optimization
# Homework 2

Due: September 28, 2020

September 21, 2020

**1.** Prove. If $C_1 \subseteq R^m$ and $C_2 \subseteq R^n$ are convex, so is $C_1 \times C_2 = \{(x, y) \subseteq R^{m+n} | x \in C_1, y \in C_2\}$.

**2.** Let $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$ be an affine function from $\mathbb{R}^n$ to $\mathbb{R}^m$, where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. Show that if $D \subset \mathbb{R}^m$ is convex, so is its inverse image $f^{-1}(D) \triangleq \{\boldsymbol{x} : f(\boldsymbol{x}) \in D\}$.

**3.** Show by induction on $m$ that if $C \subset \mathbb{R}^n$ is convex and $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m \in C$, then any convex combination $\sum_{i=1}^m \theta_i \boldsymbol{x}_i \in C$. (We have sketched the proof. Please fill in the details.)

**4.** Prove the convex hull $\operatorname{conv} S$ of $S$ is the set of all convex combinations of points in $S$ by completing the following steps. You can assume the result in Problem 3.

(a). Let

$$C = \left\{ \sum_{i=1}^m \theta_i \boldsymbol{x}_i : m \in \mathbb{N}; \boldsymbol{x}_i \in S, \theta_i \geq 0, i = 1, \ldots, m; \sum_{i=1}^m \theta_i = 1 \right\}$$

Show that $C$ is convex.

(b). Show that $C \subset \operatorname{conv} S$ and conclude $C = \operatorname{conv} S$.

**5.** Let $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_K$ be distinct points in $\mathbb{R}^n$. Let $V$ be the set of points that are closer in Euclidean distance to $\boldsymbol{x}_0$ than $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_K$, called the **Vonoroi region** around $\boldsymbol{x}_0$ with respect to $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_K$, i.e.

$$V = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{x}_0\|_2 \leq \|\boldsymbol{x} - \boldsymbol{x}_i\|_2, i = 1, 2, \ldots, K\}.$$

Show that $V$ is a polyhedron by identifying $\boldsymbol{A}$ and $\boldsymbol{b}$ such that $V = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$. (You don't have to draw it in the submission, but try to visualize $V$ for $\mathbb{R}^2$.)