CS257 Linear and Convex Optimization

Homework 4

Due: October 19, 2020

October 12, 2020

1. Determine if the following functions are convex, concave, or neither.

(a). $f(\boldsymbol{x}) = f(x_1, x_2, x_3) = 2x_1^2 + x_1x_3 + x_2^2 + 2x_2x_3 + \frac{1}{2}x_3^2$ on $\mathbb{R}^3$

(b). $f(\boldsymbol{x}) = f(x_1, x_2) = (x_1x_2)^{-1}$ on $\mathbb{R}^2_{++} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$

(c). $f(x_1, x_2) = x_1x_2$ on $\mathbb{R}^2_{++} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$

(d). $f(x_1, x_2) = \frac{x_1}{x_2}$ on $\mathbb{R}^2_{++} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$

(e). $f(x_1, x_2) = x_1^{\alpha}x_2^{1-\alpha}$, where $0 \le \alpha \le 1$, on $\mathbb{R}^2_{++} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$

2. Prove the following statements.

(a). $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^4$ is strictly convex over $\mathbb{R}$.

(b). $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x_1, x_2) = x_1^2 + x_2^4$ is strictly convex over $\mathbb{R}^2$.

Hint: Use the first-order condition.

3. Suppose $f : \mathbb{R}^n \to (-\infty, +\infty]$ satisfies Jensen's inequality

$$f(\theta\boldsymbol{x} + \bar{\theta}\boldsymbol{y}) \le \theta f(\boldsymbol{x}) + \bar{\theta} f(\boldsymbol{y}), \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, \theta \in [0, 1]$$

Show that the domain of f

$$\operatorname{dom} f = \{\boldsymbol{x} \in \mathbb{R}^n : f(\boldsymbol{x}) < \infty\}$$

is convex.

4. Is the following set convex? Show your argument.

$$S = \{\boldsymbol{x} \in \mathbb{R}^2 : \boldsymbol{x} > \boldsymbol{0}, x_1 \log x_1 + x_2 \log x_2 \le 2\}$$

5. Determine whether the following optimization problems are convex optimization or not. Give your reasons.

(a).

$$\begin{aligned} \min_{x_1,x_2} \quad & x_1^2 - 2x_1x_2 + x_2^2 + x_1 + x_2 \\ \text{s.t.} \quad & (x_1 - x_2)^2 + 4x_1x_2 + e^{x_1+x_2} \leq 0 \\ & x_1 - 3x_2 = 0 \end{aligned}$$

(b).

$$\begin{aligned} \min_{x_1,x_2} \quad & x_1^2 + x_2^4 \\ \text{s.t.} \quad & x_1 e^{-(x_1+x_2)} \leq 0 \\ & x_1^2 - 2x_1x_2 + x_2^2 + x_1 + x_2 \leq 0 \\ & 6x_1^2 - 7x_2 = 0 \end{aligned}$$