CS257 Linear and Convex Optimization

Homework 5

Due: October 26, 2019

October 19, 2019

Please also submit all your python source codes in separate .py files on Canvas.

1. Convert the following LP problem into standard form.

$$\begin{aligned} \max_{x_1,x_2} \quad & 2x_1 - 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 = 7 \\ & x_1 - 2x_2 \le 4 \\ & x_1 \ge 0 \end{aligned}$$

2. Consider the optimization problem

$$\begin{aligned} \min_{x_1,x_2} \quad & f(x_1, x_2) \\ \text{s.t.} \quad & 2x_1 + x_2 \ge 1 \\ & x_1 + 3x_2 \ge 1 \\ & x_1 \ge 0, x_2 \ge 0 \end{aligned}$$

Sketch the feasible set. For the objective functions in (a)-(c), first find the set of optimal solutions and the optimal value graphically, and then solve them using CVXPY. For the objective functions in (d) and (e), solve them using CVXPY. Show the outputs of CVXPY.

(a). $f(x_1, x_2) = x_1 + x_2$

(b). $f(x_1, x_2) = -x_1 - x_2$

(c). $f(x_1, x_2) = x_1$

(d). $f(x_1, x_2) = \max\{x_1, x_2\}$

(e). $f(x_1, x_2) = x_1^2 + 9x_2^2$

3. Consider the following optimization problem with $\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{b} \in \mathbb{R}^m$.

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_1 \\ \text{s.t.} \quad & \|\boldsymbol{x}\|_\infty \le 1 \end{aligned} \tag{1}$$

(a). Reformulate problem (1) as an LP.

(b). Solve the original problem in (1) using CVXPY for $m = 3$, $n = 2$,

$$\boldsymbol{A} = \begin{pmatrix} 3 & 2 \\ 2 & -4 \\ 2 & 5 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 10 \\ 12 \\ -6 \end{pmatrix}.$$

Show the output of CVXPY.

(c). Solve the LP reformulation you find in part (a) using CVXPY. Show the output of CVXPY.

4. Let $\boldsymbol{w} \in \mathbb{R}^2$ and

$$\boldsymbol{X} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}.$$

Solve problem (a) by hand and problems (b), (c) with CVXPY and show the outputs.

(a). Linear least squares regression

$$\min_{\boldsymbol{w}} \|\boldsymbol{X}\boldsymbol{w} - \boldsymbol{y}\|_2^2$$

(b). Lasso

$$\begin{aligned} \min_{\boldsymbol{w}} \quad & \|\boldsymbol{X}\boldsymbol{w} - \boldsymbol{y}\|_2^2 \\ \text{s.t.} \quad & \|\boldsymbol{w}\|_1 \le t \end{aligned}$$

for $t = 1$ and $t = 10$. In each case, is the solution the same as that of (a)? Is the solution sparse? Here 'sparse' means that, at a solution $\boldsymbol{w}^*$, we will have most components of $\boldsymbol{w}^*$ close to zero. (Note that there are numerical errors, which you should ignore when answering the questions.)

(c). Ridge regreesion

$$\begin{aligned} \min_{\boldsymbol{w}} \quad & \|\boldsymbol{X}\boldsymbol{w} - \boldsymbol{y}\|_2^2 \\ \text{s.t.} \quad & \|\boldsymbol{w}\|_2^2 \le t \end{aligned}$$

for $t = 1$ and $t = 100$. In each case, is the solution the same as that of (a)? Is the solution sparse? (Again ignore the numerical errors.)

Remark. To understand why Lasso produces sparse solutions, draw a contour map of the objective function and the feasible region in the $\boldsymbol{w}$-plane; see also Figure 2 of the Lasso paper http://statweb.stanford.edu/~tibs/lasso/lasso.pdf