EI331 Signals and Systems

Homework 2

Due: Thursday, March 14

March 7, 2019

1. Recall that for a complex function of a real variable $f(t) = u(t) + jv(t)$, we have

$$f'(t) = u'(t) + jv'(t).$$

Also recall

$$e^{ct} = e^{\sigma t}[\cos(\omega t) + j\sin(\omega t)]$$

for $c = \sigma + j\omega$. Show that

$$\frac{d}{dt}e^{ct} = ce^{ct}.$$

2. Recall that for a complex function of a real variable $f(t) = u(t) + jv(t)$, we have

$$\int_a^b f(t)dt = \int_a^b u(t)dt + j\int_a^b v(t)dt.$$

Use Problem 1 to show that for $a, b \in \mathbb{R}$ and $c \in \mathbb{C}$,

$$\int_a^b e^{ct}dt = \frac{e^{cb} - e^{ca}}{c}.$$

Remark. Problems 1 and 2 show that we can differentiate and integrate e^{ct} for $c \in \mathbb{C}$ as we do for $c \in \mathbb{R}$.

3. (OWN 1.12) Determine the values of the integers M and n_0 so that the following equation holds

$$1 - \sum_{k=3}^{\infty} \delta[n-1-k] = u[Mn - n_0]$$

4. Recall

$$r_\Delta(t) = \frac{u(t+\frac{\Delta}{2}) - u(t-\frac{\Delta}{2})}{\Delta}.$$

Let

$$u_1(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

(a). Compute $\int_{\mathbb{R}} u_1(t) r_\Delta(t)dt$.

(b). Is $\lim_{\Delta \to 0} \int_{\mathbb{R}} u_1(t) r_\Delta(t) = u_1(0)$ true?

Remark. The exercise shows why we require continuity of $x(t)$ at $t = 0$ in the sampling property

$$\int_{\mathbb{R}} x(t)\delta(t)dt = x(0).$$

Similarly, we need $x(t)$ to be continuous at $t = a$ in the more general formula

$$\int_{\mathbb{R}} x(t)\delta(t-a)dt = x(a).$$

5. Recall that for $a < b$,

$$\int_a^b x(t)\delta(t)dt \triangleq \int_{\mathbb{R}} x(t)\delta(t)[u(t-a) - u(t-b)]dt, \tag{1}$$

where u is the CT unit step function. Show that

$$\int_a^b x(t)\delta(t)dt = \begin{cases} x(0), & \text{if } a < 0 < b, \\ 0, & \text{if } a > 0 \text{ or } b < 0. \end{cases}$$

Remark. The integral $\int_{\mathbb{R}} u(t)\delta(t)dt$ is not defined since u is discontinuous at $t = 0$, so the integral in (1) is not well defined if either $a = 0$ or $b = 0$. You may have seen the following formula in some books

$$\int_{0-}^{0+} x(t)\delta(t)dt = x(0),$$

where the left-hand side simply means

$$\lim_{a \uparrow 0, b \downarrow 0} \int_a^b x(t)\delta(t)dt.$$

More generally, we have

$$\int_a^b x(t)\delta(t-c)dt = \begin{cases} x(c), & \text{if } a < c < b, \\ 0, & \text{if } a > c \text{ or } b < c. \end{cases}$$

6. Evaluate the following expressions

(a). $\int_0^1 \sin(2t-3)\delta(1-2t)dt$

(b). $x'(t)$, where $x(t) = e^{(-1+j2)t}u(t-1)$

(c). $x'(t)$, where $x(t) = (t+1)u(t) + tu(-t)$ (Hint: You can assume $u(0) = 1/2$.)

7. (OWN 1.17(a)) Is the CT system $y(t) = x(\sin(t))$ causal?

8. Let x be a DT periodic signal with period N. Let $y = T(x)$ be the output of a time-invariant system T. Show that y is also periodic with period N.

9. (OWN 1.27) Determine whether the following systems are memoryless, time invariant, linear, causal, and stable. Justify your answers. Assume $u(0) = 1$.

(a). $y(t) = x(t-2) + x(2-t)$

(b). $y(t) = [x(t) + x(t-2)]u(t)$

(c). $y(t) = u[x(t) + x(t-2)]$

10. **(Bonus)** Recall a family of kernels $\{K_\Delta(t)\}_{\Delta>0}$ is called a family of *good kernels* or an *approximation to the identity* if the following conditions hold,

(1). for all $\Delta > 0$, $\int_{-\infty}^{\infty} K_\Delta(t)dt = 1$;

(2). for some $M > 0$ and all $\Delta > 0$, $\int_{-\infty}^{\infty} |K_\Delta(t)|dt < M$;

(3). for every $\eta > 0$, $\lim_{\Delta\to 0} \int_{|t|>\eta} |K_\Delta(t)|dx = 0$.

Suppose $x(t)$ is continuous at $t = 0$ and $\|x\|_\infty = \sup_{t\in\mathbb{R}} |x(t)| < \infty$. Show that

$$I_\Delta \triangleq \int_{\mathbb{R}} x(t)K_\Delta(t)dt \to x(0) \qquad \text{as } \Delta \to 0,$$

by completing the following steps.

(a). Show $|I_\Delta - x(0)| \le \int_{\mathbb{R}} |x(t) - x(0)| \cdot |K_\Delta(t)|dt$.

(b). Show that given any $\epsilon > 0$, there exists some $\eta > 0$ such that

$$J \triangleq \int_{|t|\le\eta} |x(t) - x(0)| \cdot |K_\Delta(t)|dt < \epsilon M.$$

(c). Show that there exists some $B > 0$ such that

$$J_\Delta \triangleq \int_{|t|>\eta} |x(t) - x(0)| \cdot |K_\Delta(t)|dt < 2B \int_{|t|>\eta} |K_\Delta(t)|dx.$$

(d). Show that for any $\epsilon > 0$,

$$\lim_{\Delta\to 0} |I_\Delta - x(0)| \le \epsilon M.$$

(e). Conclude that

$$\lim_{\Delta\to 0} |I_\Delta - x(0)| = 0$$

and

$$\lim_{\Delta\to 0} I_\Delta = x(0).$$

Remark. This exercise shows that K_Δ indeed converges to the unit impulse function as $\Delta \to 0$, provided that x is also bounded.