EI331 Signals and Systems
Homework 3

Due: Thursday, March 21

March 14, 2019

1. Recall the support of a DT signal x is $\operatorname{supp} x = \{n \in \mathbb{Z} : x[n] \neq 0\}$.

(a). Suppose $\operatorname{supp} x_i \subset [a_i, b_i]$ for some integers a_i, b_i, $i = 1, 2$. Show $\operatorname{supp}(x_1 * x_2) \subset [a_1 + a_2, b_1 + b_2]$.

(b). Find the supports of $x[n] = u[n] - u[n-2]$ and $x * x$. Can you improve the result in (a) for general x_1 and x_2, i.e. can you find a, b such that $\operatorname{supp}(x_1 * x_2) \subset [a, b]$ and either $a > a_1 + a_2$ or $b < b_1 + b_2$?

2. Denote by $\mathcal{R}$ the set of right-sided DT signals, i.e.

$$\mathcal{R} = \left\{x \in \mathbb{C}^{\mathbb{Z}} : \exists N(x) > -\infty \text{ such that } x[n] = 0 \text{ for } n < N(x)\right\}.$$

Consider two DT signals $x_1, x_2 \in \mathcal{R}$. Suppose $x_i[n] = 0$ for $n < N(x_i)$, $i = 1, 2$. Show that $(x_1 * x_2)[n] = 0$ if $n < N(x_1) + N(x_2)$. Conclude that $\mathcal{R}$ is closed under convolution, i.e. $x_1, x_2 \in \mathcal{R}$ implies $x_1 * x_2 \in \mathcal{R}$.

3. Denote by ℓ_p the set of DT signals with finite ℓ_p norm, i.e.

$$\ell_p = \left\{ x \in \mathbb{C}^{\mathbb{Z}} : \|x\|_p = \left(\sum_{n=-\infty}^{\infty} |x[n]|^p \right)^{1/p} < \infty \right\}.$$

(a). Consider two DT signals $x_1, x_2 \in \ell_1$. Show

$$\|x_1 * x_2\|_1 \leq \|x_1\|_1 \cdot \|x_2\|_1.$$

Conclude that ℓ_1 is closed under convolution, i.e. $x_1, x_2 \in \ell_1$ implies $x_1 * x_2 \in \ell_1$

(b). Consider two DT signals $x_1, x_2 \in \ell_2$. Show

$$\|x_1 * x_2\|_\infty \leq \|x_1\|_2 \cdot \|x_2\|_2.$$

You can assume the following Cauchy-Schwarz inequality, which we will see later

$$\left| \sum_k x[k] y[k] \right| \leq \|x\|_2 \cdot \|y\|_2.$$

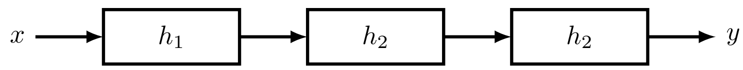
4. Compute $x * h$, where

$$x[n] = \left(\frac{1}{3}\right)^{n-1} u[n-2]$$

and

$$h[n] = u[n+3]$$

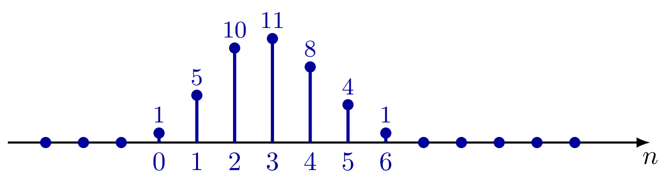
5. (OWN 2.24) Consider the following cascaded causal LTI systems.



The impulse response h_2 is

$$h_2[n] = u[n] - u[n-2]$$

The nonzero values of the overall impulse response is shown below.



(a). Find the impulse response h_1.

(b). Find the overall response of the system to the input $x[n] = \delta[n] - \delta[n-1]$.

6. Let $x_i(t) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{(t-\mu_i)^2}{2\sigma_i^2}\right)$.

(a). Compute $x_1 * x_2$. You can use the following identity

$$\int_{\mathbb{R}} e^{-a^2 z^2} dz = \sqrt{\frac{\pi}{a^2}}.$$

(b). Recall from probability theory that x_i is the density of a Gaussian distribution with mean μ_i and variance σ_i^2 and that the density of the sum of two independent random variables is given by the convolution of their densities. What does the calculation in (a) say about the distribution of the sum of two independent Gaussian random variables?

7. Define the n-fold convolution of x recursively by

$$x^{*n} = \begin{cases} x, & n = 1 \\ x * x^{*(n-1)}, & n \geq 2 \end{cases}$$

Let $x(t) = u(t+T) - u(t-T)$, where $T > 0$. Recall that $(x * x)(t) = (2T - |t|)[u(t+2T) - u(t-2T)]$. Compute x^{*3}.

8. (OWN 2.40) Consider an LTI system with input-output relation

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau - 2) d\tau$$

(a). What's the impulse response of the system?

(b). Is the system causal?

(c). Is the system stable?

(d). What's the response when the input is $x(t) = u(t+1) - u(t-2)$?

9. Consider the system described by the following ODE,

$$y'(t) + 3y(t) = x(t).$$

(a). If the system is at initial rest, what's the response when the input is $x(t) = e^{2t}u(t-1)$?

(b). If the system is at initial rest, what's the response when the input is $x(t) = e^{-3t}u(t)$?

(c). If the system initial state is $y(0) = 2$, what's the zero-input response?