# EI331 Signals and Systems
# Homework 4

Due: Thursday, March 28

March 23, 2019

**1.** Let

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

(a). Compute $B^n$ for $n \geq 2$.($B$ is an example of a **nilpotent matrix** of **index** 3.)

(b). Show

$$A^n = \lambda^n I + n\lambda^{n-1} B + \frac{n(n-1)}{2}\lambda^{n-2} B^2$$

You can either use induction on $n$ or assume the following binomial theorem for matrices: If $A_1$ and $A_2$ commute, i.e. $A_1 A_2 = A_2 A_1$, then

$$(A_1 + A_2)^n = \sum_{k=0}^{n} \binom{n}{k} A_1^k A_2^{n-k}.$$

(c). Show

$$e^{At} = e^{\lambda t} I + t e^{\lambda t} B + \frac{1}{2} t^2 e^{\lambda t} B^2 \tag{1}$$

**2.** Recall the following third-order ODE

$$y''' + a_2 y'' + a_1 y' + a_0 y = x. \tag{2}$$

can be converted into the following equivalent first-order ODEs

$$Y' = AY + bx,$$

where $Y = (Y_0, Y_1, Y_2)^T \triangleq (y, y', y'')$, and

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(a). Find the characteristic equation $p(\lambda) = \det(\lambda I - A)$ of the matrix $A$

(b). What's the relationship between $p(\lambda)$ and the characteristic equation of the ODE (2)?

(c). Assume $a_0 = 1$, $a_1 = a_2 = 3$ for (c)–(g). Show that Eq. (2) has only one distinct characteristic root $\lambda = -1$ of multiplicity 3.

(d). Using the fact $A = P\Lambda P^{-1}$, where

$$\Lambda = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 3 & 6 \\ -1 & -2 & -3 \\ 1 & 1 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 3 & 3 \\ -2 & -5 & -3 \\ 1 & 2 & 1 \end{pmatrix},$$

find the matrices $B_1$ and $B_2$ in the following identity

$$e^{At} = e^{-t}I + te^{-t}B_1 + t^2e^{-t}B_2.$$

Hint: Use (1).

(e). Find the zero-input response of (2) with initial condition $y(0) = y'(0) = 0$ and $y''(0) = 1$

(f). Find the impulse response of (2) with the initial rest condition.

(g). Find the response of (2) for $x(t) = \delta'(t) + \delta''(t)$ with the initial rest condition.

**Remark.** You should convince yourselves that the conclusion in (b) holds for higher-order ODEs as well. From part (d), can you see why homogeneous solutions of (2) are linear combinations of $e^{-t}$, $te^{-t}$ and $t^2e^{-t}$? Similarly, you can show why homogeneous solutions of linear constant-coefficient difference equations take the form given in class.

**3.** (OWN 2.11) Let $x(t) = u(t-3) - u(t-5)$ and $h(t) = e^{-3t}u(t)$.

(a). Compute $y = x * h$.

(b). Compute $g = x' * h$

(c). How is $g$ related to $y$?

**4.** (OWN 2.19) Consider the cascade of two causal LTI systems as shown below.

$$x[n] \longrightarrow \boxed{\;S_1\;} \xrightarrow{w[n]} \boxed{\;S_2\;} \longrightarrow y[n]$$

$S_1$ and $S_2$ are described by the following difference equations

$$S_1: \quad w[n] - \frac{1}{2}w[n-1] = x[n]$$

$$S_2: \quad y[n] - \alpha y[n-1] = \beta w[n]$$

The difference equation relating $x$ and $y$ is

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n]$$

(a). Find the values of $\alpha$ and $\beta$.

(b). Find the overall impulse response of the cascaded system.

**5.** Consider a system described by an $N$-th order linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = x[n]$$

with the initial rest condition. Let $y_1[n]$ be the response to $x_1[n] = x_2[n]u[n]$, where $u$ is the DT unit step function. Recall that $y_1[n] = 0$ for $n < 0$. To find $y_1[n]$ for $n \geq 0$, we need to solve

$$\begin{cases} \sum_{k=0}^{N} a_k y_1[n-k] = x_1[n] = x_2[n], \ n \geq 0 \\ y_1[k] = 0, \quad k = -1, -2, \ldots, -N \end{cases} \tag{3}$$

Let $y_2$ be the solution to the following initial value problem

$$\begin{cases} \sum_{k=0}^{N} a_k y_2[n-k] = x_2[n], \ \forall n \in \mathbb{Z} \\ y_2[k] = 0, \quad k = -1, -2, \ldots, -N \end{cases} \tag{4}$$

Use the iterative method and induction on $n$ to show that $y_1[n] = y_2[n]$ for $n \geq 0$

**Remark.** This result shows that in order to find the solution $y_1$ to (3), we can first find the solution $y_2$ to (4) and then set $y_1 = y_2 u$. This is convenient when we use the decomposition into particular and homogeneous solutions, because we can use the values $y[-1], \ldots, y[-N]$ directly to determine the coefficients in $y_1$ using the expression that is valid only for $n \geq 0$. Alternatively, you can use the iterative method to figure out $y[0], y[1], \ldots, y[N-1]$ and use this set of values to determine the coefficients.

**6.** Consider the system described by the following difference equation,

$$y[n] - 3y[n-1] + 2y[n-2] = x[n].$$

Assume the system is at initial rest. Find the response to the input $x[n] = (1 + 2^n + 3^n)u[n]$.

**7.** (OWN 2.46) Consider an LTI system $T$ and a signal $x(t) = 2e^{-3t}u(t-1)$. If $y = T(x)$ and $T(x') = -3y + e^{-2t}u(t)$, find the impulse response $h$ of $T$. Is this system causal?

**8.** Express $\sin(t)\delta''(t)$ as a linear combination of derivatives of $\delta$ with constant coefficients.