EI331 Signals and Systems
Homework 6

Due: Thursday, April 11

April 5, 2019

1. In this problem, we show that not all norms can be induced by an inner product.

(a). Show that in an inner product space V with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, the follow **parallelogram law** holds for any $x, y \in V$

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

(You should be able to visualize what this means geometrically when $V = \mathbb{R}^2$.)

(b). Show that the parallelogram law fails for 1-norm by considering $x = (1,0)^T$ and $y = (0,1)^T$ in $\mathbb{R}^2$. Conclude that 1-norm on $\mathbb{R}^2$ cannot be induced by an inner product.

Remark. The parallelogram law turns out to be also sufficient for a norm to be the induced norm of an inner product. In fact, you can verify that the following so-called **polarization identity** defines an inner product in terms of the norm

$$\langle x, y \rangle = \frac{1}{4} \sum_{n=0}^{3} j^n \|x + j^n y\|^2$$

2. (OWN 3.22) Find the Fourier coefficients of the CT periodic signals in Figure 1. Hint: use the differentiation and integration properties.

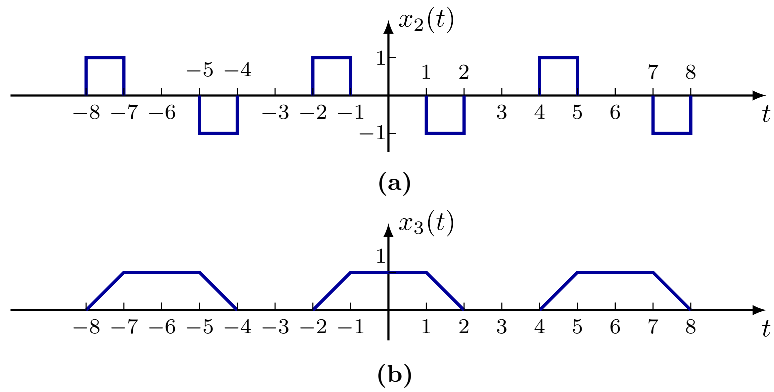

Figure 1: Problem 2.

3. (OWN 3.11) Suppose we are given the following information about a DT signal $x[n]$,

(a). $x[n]$ is even

(b). $x[n]$ has period $N = 10$ and Fourier coefficients $\hat{x}[k]$

(c). $\hat{x}[11] = 5$

(d). $\frac{1}{10}\sum_{n=0}^{9} |x[n]|^2 = 50$

Show that $x[n] = A\cos(Bn + C)$ and find the numerical values of A, B and C.

4. (OWN 3.14) Recall the system function of a DT system with impulse response $h[n]$ is

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}.$$

The frequency response is obtained when $z = e^{j\omega}$, i.e.

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n},$$

which is the eigenvalue of the system associated with the eigenfunction $e^{j\omega n}$. When the DT impulse train

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - 4k]$$

is the input to a particular LTI system with frequency response $H(e^{j\omega})$, the output of the system is found to be

$$y[n] = \cos\left(\frac{5\pi}{2}n + \frac{\pi}{4}\right).$$

Determine the values of $H(e^{jk\pi/2})$ for $k = 0, 1, 2, 3$.

5. (OWN 3.12) Consider two DT signals x_1, x_2 with period $N = 4$. Their Fourier coefficients are

$$\hat{x}_1[0] = \hat{x}_1[3] = \frac{1}{2}\hat{x}_1[1] = \frac{1}{2}\hat{x}_1[2] = 1$$

$$\hat{x}_2[0] = \hat{x}_2[1] = \hat{x}_2[2] = \hat{x}_2[3] = 1$$

Use the multiplication property to find the Fourier coefficients of $y[n] = x_1[n]x_2[n]$.

6. Given a finite sequence $x = (x[0], x[1], \dots, x[N-1])$, its **discrete Fourier transform (DFT)** is defined by

$$X[k] = \mathrm{DFT}(x)[k] = \sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n}, \quad k = 0, 1, 2, \dots, N-1.$$

The **inverse discrete Fourier transform (IDFT)** is given by

$$x[n] = \mathrm{IDFT}(X)[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{jk\frac{2\pi}{N}n}, \quad n = 0, 1, 2, \dots, N-1.$$

The sequence x can be considered as one period of a periodic signal x_N. Another way to say this is that we can extend x periodically with period N, i.e. define $x_N[n + rM] = x[n]$, for $n = 0, 1, \ldots, N-1$ and $r \in \mathbb{Z}$,

(a). How is the DFT X of x related to the DT Fourier series coefficients $\widehat{x_N}$ of x_N?

(b). Given two finite sequences x, y of length N, the so-called **circular convolution** $x \circledast_N y$ of x and y is defined by

$$(x \circledast_N y)[n] = (x_N * y_N)[n], \quad n = 0, 1, \ldots, N-1.$$

where $x_N * y_N$ is the **periodic convolution** of x_N and y_N.

Show the following identity using part (a) and the property of Fourier series coefficient for periodic convolution,

$$\mathrm{DFT}(x \circledast_N y) = \mathrm{DFT}(x)\,\mathrm{DFT}(y)$$

(c). DFT and IDFT can be computed using a fast algorithm called **Fast Fourier Transform (FFT)**. For example, you can compute $\mathrm{DFT}(x)$ and $\mathrm{IDFT}(X)$ using the python library `numpy`,

```
X = numpy.fft.fft(x)
x = numpy.fft.ifft(X)
```

The circular convolution $x \circledast y$ can be computed using $x \circledast_N y = \mathrm{IDFT}(\mathrm{DFT}(x)\,\mathrm{DFT}(y))$. Use `numpy` (or your favorite programming language; do **not** use Matlab, which seems to have a bug for FFT) to compute the circular convolution of `x = [2,-2,-2,-1,0]` and `y=[-3,-3,-2,1,-1]` with $N = 5$.

(d). The finite sequence x can also be naturally considered as an infinite sequence, also denoted by x, by assuming $x[n] = 0$ for $n < 0$ and $n \geq N$. Then we can define the **aperiodic convolution** of two finite sequences x and y in the usual way, i.e.

$$(x * y)[n] \triangleq \sum_{m=-\infty}^{\infty} x[m]y[n-m] = \sum_{m=0}^{N-1} x[m]y[n-m]$$

Compute $x * y$ using `numpy.convolve` for the two sequences in part (c); see Lecture 5 for an example.

(e). Note that the periodic extension x_N is related to the infinite sequence x by the following

$$x_N[n] = \sum_{r=-\infty}^{\infty} x[n - rN].$$

Given two finite sequences of length N, show

$$(x \circledast_N y)[n] = \sum_{r=-\infty}^{\infty} (x * y)[n - rN], \quad n = 0, 1, \ldots, N-1.$$

You should be able to verify this relation using the results in parts (c) and (d).

(f). Part (e) shows that in general $x \circledast_N y \neq x * y$. In order to leverage the fast algorithm FFT to compute $x*y$, we can do the following. Recall from Problem 1 of Homework 3, $\mathrm{supp}(x*y) \subset [0, 2N-2]$. Instead of extending x with period N, we extend it with period $M \geq 2N-1$, i.e.

$$x_M[n] = \sum_{r=-\infty}^{\infty} x[n-rM].$$

In one period,

$$x_M[n+rM] = \begin{cases} x[n], & n = 0, 1, \ldots, N-1 \\ 0, & n = N, \ldots, M-1 \end{cases}$$

This is called **zero padding**. We can compute the circular convolution of x and y with period M,

$$(x \circledast_M y)[n] = (x_M * y_M)[n], \quad n = 0, 1, \ldots, M-1.$$

The same reasoning in (e) (you don't have to write it down, but think about it) gives

$$(x \circledast_M y)[n] = \sum_{r=-\infty}^{\infty} (x * y)[n-rM], \quad n = 0, 1, \ldots, M-1.$$

Since $\text{supp}(x * y) \subset [0, 2N-2]$ and $M \geq 2N-2$, it follows that

$$(x \circledast_M y)[n] = (x * y)[n], \quad n = 0, 1, \ldots, 2N-2.$$

Use `fft` and `ifft` in `numpy` to compute the aperiodic convolution of x and y in part (c). You should recover the result in (d) up to some numerical errors.

You should email us your source code for parts (c), (d) and (f).