EI331 Signals and Systems

Lecture 1

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1. Definition of Signals

2. Transformations of Independent Variable

3. Some Properties of Signals
3.1 Energy and power
3.2 Periodicity
3.3 Even/Odd symmetry

Signals

Definition [The American Heritage® Dictionary of the English Language]

a. *Electronics* An impulse or fluctuating quantity, as of electrical voltage or light intensity, whose variations represent coded information.
b. *Computers* A sequence of digital values whose variations represent coded information.

Examples

- voltages or currents in circuits
- images, videos

Mathematical Representation

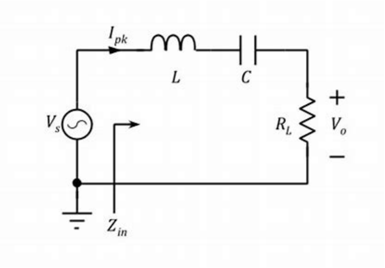
Function of one or more independent variables

$$\begin{aligned} x : I &\to X \\ t &\mapsto x(t) \end{aligned}$$

Examples of Signals

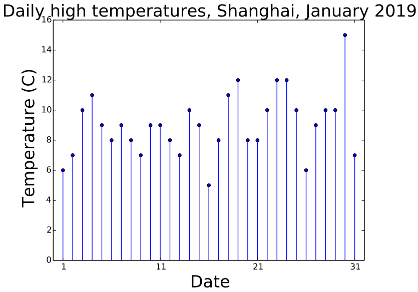
Electrical voltage

$$V_o : \mathbb{R} \to \mathbb{R}$$
$$t \mapsto V_o(t)$$



Daily temperature

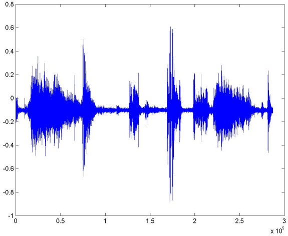
$$T : I \to \mathbb{R}$$
$$n \mapsto T[n]$$



Examples of Signals

Speech signal

$$x : \mathbb{R} \to \mathbb{R}$$
$$t \mapsto x(t)$$



Color Image

$$P : I \times J \to R \times G \times B$$
$$(i,j) \mapsto (r[i,j], g[i,j], b[i,j])$$

Continuous-time vs. Discrete-time Signals

Focus on signals of 1-D independent variable

- $x : I \to X$, with $I \subset \mathbb{R}$, often $X \subset \mathbb{R}$ or $\mathbb{C}$
- independent variable often referred to as "time"

Continuous-time (CT) signal: $x(t)$

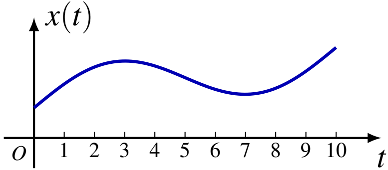
- defined for interval $I \subset \mathbb{R}$, often $I = \mathbb{R}$
- called analog signal if X is also continuum
- notation: parentheses for continuous time, e.g. (t)

Discrete-time (DT) signal: $x[n]$

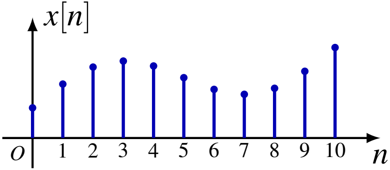
- defined for discrete set I, often $I = \mathbb{Z}$
- called digital signal if X is also discrete
- notation: square brackets for discrete time, e.g. $[n]$

Continuous-time vs. Discrete-time Signals

CT signal



DT signal



Signals from physical systems often continuous-time

- electrical current, car speed

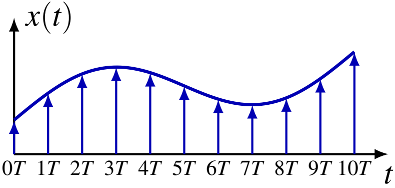
Signals from computation systems often discrete-time

- mp3, digital image

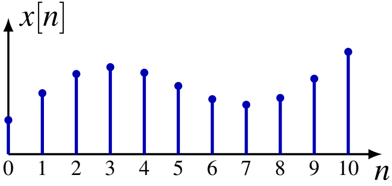
Continuous-time vs. Discrete-time Signals

Sampling: converts CT signals to DT signals

CT signal



DT signal



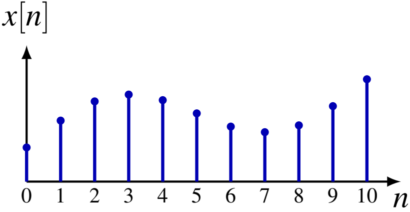
T = sampling period

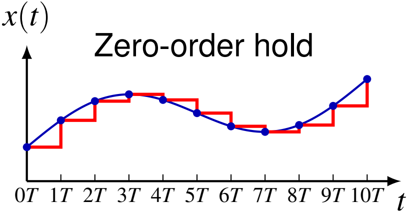
Important for computer processing of physical signals

- sampled data contains no information about T
- uniform sampling most common

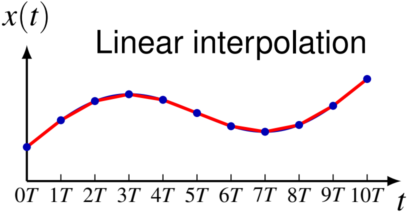
Continuous-time vs. Discrete-time Signals

Reconstruction: converts DT signals to CT signals





T = sampling period



Different T yields different reconstructed signals

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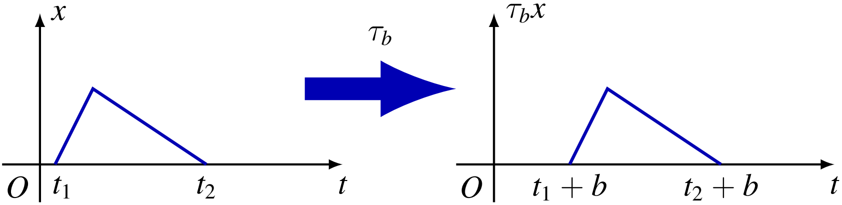
Time Shift

Time shift (Translation) operator $\tau_b : x \mapsto \tau_b x$

$$(\tau_b x)(t) = x(t-b) \qquad (\tau_b x)[n] = x[n-b]$$

$$b \in \mathbb{R} \qquad b \in \mathbb{Z}$$



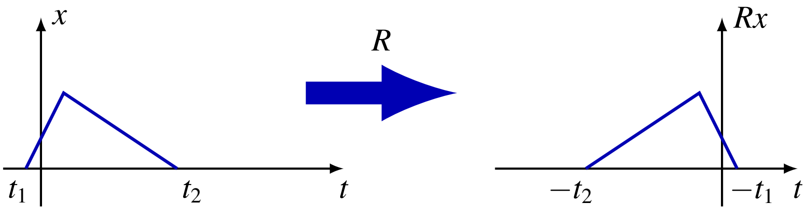
Example: Radar, sonar, radio propagation

- $b > 0$: delay by b, right shift
- $b < 0$: advance by $|b|$, left shift

Time Reversal

Time reversal (Reflection) operator $R : x \mapsto Rx$

$$(Rx)(t) = x(-t) \qquad\qquad (Rx)[n] = x[-n]$$



Example: Tape recording played backward

Time Scaling

Time scaling operator $S_a : x \mapsto S_a x$

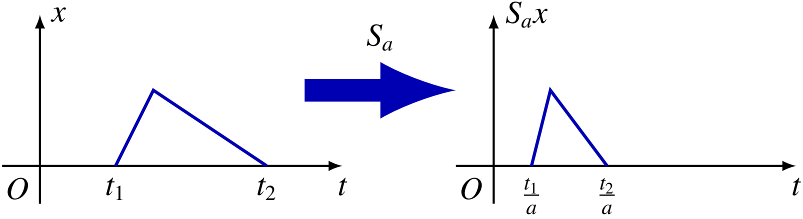
$$(S_a x)(t) = x(at)$$

$a \in \mathbb{R}_+$

$$(S_a x)[n] = x[an]$$

$a \in \mathbb{Z}_+$

need more work for $a \in \mathbb{R}_+ \setminus \mathbb{Z}_+$



Example: Audio played back at different speed

- $a > 1$: fast forward, compressed
- $0 < a < 1$: slow forward, stretched

General Affine Transformation of Time

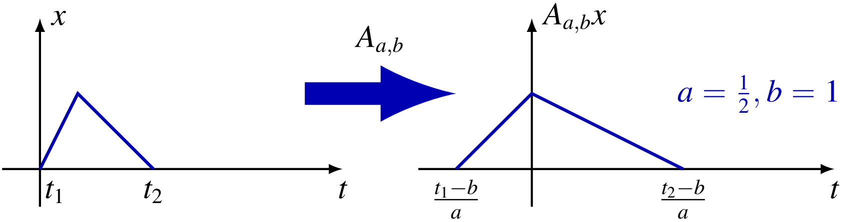
Affine transformation $A_{a,b} : x \mapsto A_{a,b}x$

$(A_{a,b}x)(t) = x(at + b)$

$a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}$

$(A_{a,b}x)[n] = x[an + b]$

$a \in \mathbb{Z} \setminus \{0\}, b \in \mathbb{Z}$



Can decompose as product of shift, reversal, scaling

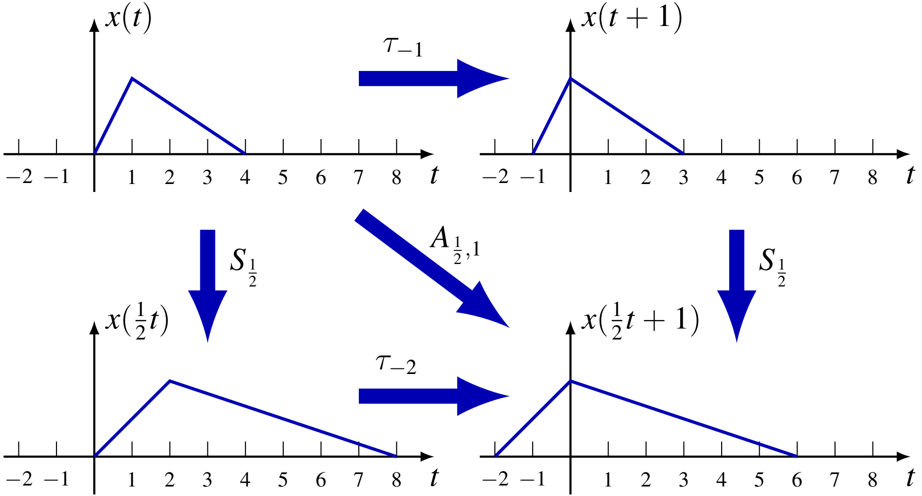
- $a > 0$: $A_{a,b} = S_a \circ \tau_{-b}$
- $a < 0$: $A_{a,b} = S_{|a|} \circ R \circ \tau_{-b}$

not unique

easier if shift first

Example of Affine Transformation

Affine transformation $A_{\frac{1}{2},1} = S_{\frac{1}{2}} \circ \tau_{-1} = \tau_{-2} \circ S_{\frac{1}{2}}$



More Identities

$$S_a \circ \tau_{-b} = \tau_{-\frac{b}{a}} \circ S_a$$

$$\begin{array}{ccc} x(t) & \xrightarrow{\tau_{-b}} & x(t+b) \\ \downarrow S_a & & \downarrow S_a \\ x(at) & \xrightarrow{\tau_{-\frac{b}{a}}} & x(at+b) \end{array}$$

$$R \circ \tau_{-b} = \tau_b \circ R$$

$$\begin{array}{ccc} x(t) & \xrightarrow{\tau_{-b}} & x(t+b) \\ \downarrow R & & \downarrow R \\ x(-t) & \xrightarrow{\tau_{b}} & x(-t+b) \end{array}$$

$$S_a \circ R = R \circ S_a$$

$$\begin{array}{ccc} x(t) & \xrightarrow{R} & x(-t) \\ \downarrow S_a & & \downarrow S_a \\ x(at) & \xrightarrow{R} & x(-at) \end{array}$$

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Signal Energy and Power

$v(t)$: voltage across 1Ω resistor

Instantaneous power $\quad p(t) = |v(t)|^2$

Energy over $[t_1, t_2]$ $\quad E(t_1, t_2) = \int_{t_1}^{t_2} |v(t)|^2 dx$

Average power over $[t_1, t_2]$ $\quad P(t_1, t_2) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |v(t)|^2 dx$

Total energy

$$E(x) = \int_{-\infty}^{\infty} |x(t)|^2 dt \qquad E(x) = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Average power

$$P(x) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \qquad P(x) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Finite-energy and Finite-power Signals

Finite-energy signal $\quad E(x) < \infty$

e.g. $\quad x(t) = 1$ for $t \in [0, 1]$ and $x(t) = 0$ elsewhere

Finite-power signal $\quad P(x) < \infty$

e.g. $\quad x(t) = \sin t$ for $t \in (-\infty, \infty)$

Some implications

- $E(x) < \infty \implies P(x) = 0$
- $P(x) > 0 \implies E(x) = \infty$

Caution

- $P(x) = 0$ does **not** imply $E(x) < \infty$
- $E(x) = \infty$ does **not** imply $P(x) > 0$

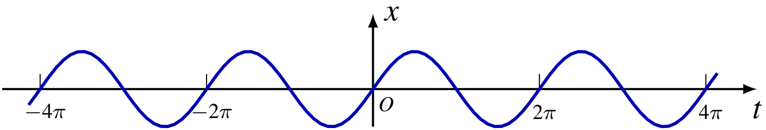
e.g. $\quad x(t) = t^{-1/2}$ for $t \geq 1$ and $x(t) = 0$ elsewhere

Periodicity: Continuous-time Signal

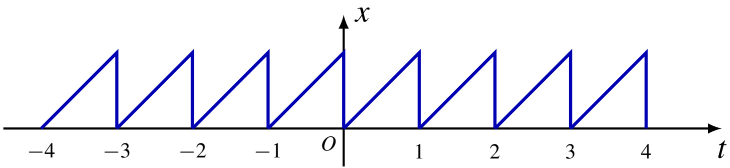
CT signal is periodic with period $T \in \mathbb{R}$ iff $\tau_T x = x$, i.e.

$$x(t+T) = x(t), \quad \forall t \in \mathbb{R}$$

Example: $x(t) = \sin t$ has period $T = 2\pi$



Example: sawtooth signal $x(t) = t - \lfloor t \rfloor$ has period $T = 1$

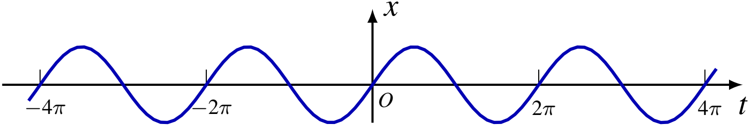


Periodicity: Continuous-time Signal

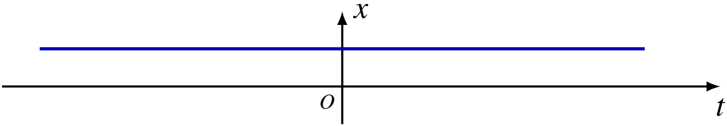
Fundamental period: smallest positive period (if exists)

$$T_0 = \min\{T > 0 : x = \tau_T x\}$$

Example: $x(t) = \sin t$ has fundamental period $T_0 = 2\pi$



Example: constant signal $x(t) = 1$ has **no** well-defined fundamental period, $\{T > 0 : x = \tau_T x\} = \mathbb{R}_+$



Periodicity: Continuous-time Signal

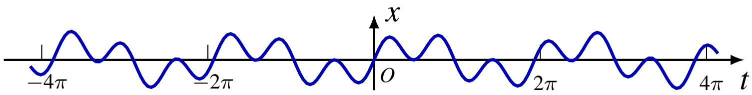
Question. What's period of $x(t) = x_1(t) + x_2(t)$ if x_i has period T_i?

Answer. **Sufficient** condition for x to have period T is $T = m_1 T_1 = m_2 T_2$ for integers m_1 and m_2. This requires

$$T_1/T_2 = m_2/m_1 \in \mathbb{Q}$$

Examples

- $x(t) = \sin(t) + \sin(2t)$has $T_0 = 2\pi$
  - $T_1 = 2\pi$, $T_2 = \pi$, $T_1/T_2 = 2$; take $m_1 = 1, m_2 = 2$.
- $x(t) = \sin(t) + \sin(3t/2)$has $T_0 = 4\pi$
  - $T_1 = 2\pi$, $T_2 = 4\pi/3$, $T_1/T_2 = 3/2$; take $m_1 = 2, m_2 = 3$.
- $x(t) = \sin(t) + \sin(\pi t)$ is **aperiodic!**



Periodicity: Continuous-time Signal

How to **prove** $x(t) = \sin(t) + \sin(\pi t)$ is aperiodic?

Proof. By contradiction. Suppose $x(t)$ has period $T > 0$.

1. $\sin(t + T) + \sin(\pi(t + T)) = \sin(t) + \sin(\pi t)$
2. $t = 1 \implies \sin(1 + T) - \sin(\pi T) = \sin(1)$
3. $t = -1 \implies \sin(-1 + T) - \sin(\pi T) = -\sin(1)$
4. subtract 3. from 2.

$$2\sin(1)\cos(T) = \sin(T + 1) - \sin(T - 1) = 2\sin(1)$$

5. 4. $\implies \cos(T) = 1 \implies T = 2k\pi$ for $k \in \mathbb{Z}_+$
6. substitute $T = 2k\pi$ into 2.

$$\sin(1) - \sin(2k\pi^2) = \sin(1) \implies \sin(2k\pi^2) = 0$$

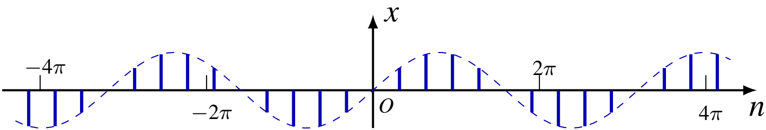
7. 6. $\implies 2k\pi^2 = m\pi$ for $m \in \mathbb{Z} \implies \pi = m/(2k) \in \mathbb{Q}$ ↯

Periodicity: Discrete-time Signal

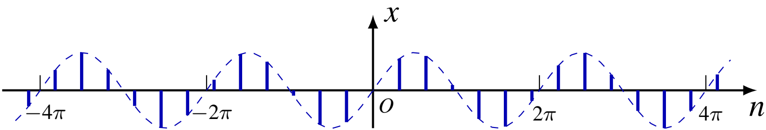
DT signal is periodic with period $N \in \mathbb{Z}$ iff $\tau_N x = x$, i.e.

$$x[n+N] = x[n], \quad \forall n \in \mathbb{Z}$$

Example: $x[n] = \sin(\frac{\pi}{5}n)$ has period $N = 10$



Example: $x[n] = \sin n$ is **aperiodic!**



Periodicity: Discrete-time Signal

Question. When is $x[n] = \sin(\omega n)$ periodic?

Solution. Suppose $x[n]$ has period $N > 0$.

$$\sin(\omega(n+N)) = \sin(\omega n) \iff \omega N = 2k\pi \text{ for } k \in \mathbb{Z}.$$

Necessary condition for periodicity

$$\frac{\omega}{2\pi} = \frac{k}{N} \in \mathbb{Q}.$$

Also sufficient (check!)

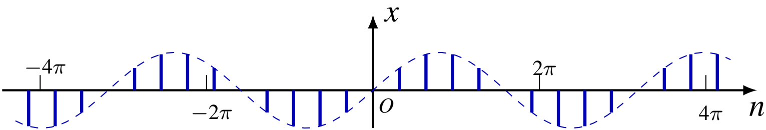
$\sin(\omega n)$ **periodic** $\iff$ ω **is rational multiple of** 2π

Periodicity: Discrete-time Signal

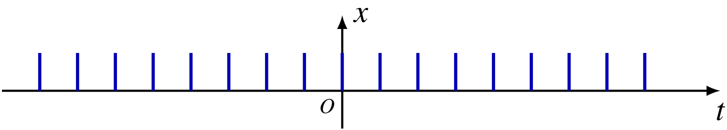
Fundamental period: smallest positive period

$$N_0 = \min\{N > 0 : x = \tau_N x\}$$

Example: $x[n] = \sin(\frac{\pi}{5}n)$ has $N_0 = 10$



Example: constant signal $x[n] = 1$ has $N_0 = 1$ **(cf.** $x(t) = 1$**!)**



Periodicity: Discrete-time Signal

Question. What's fundamental period of $x[n] = \sin(\omega n)$?

Solution. Periodic iff $\omega = 2\pi \frac{k}{N}$ for $k \in \mathbb{Z}$, $N \in \mathbb{Z}_+$. In this case, N is a period.

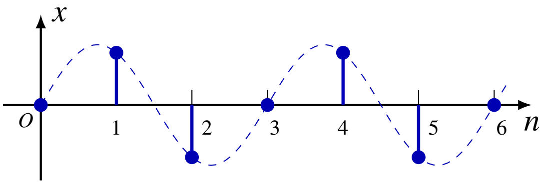
Fundamental period is $N/\gcd(N, k)$.

Proof. Clearly true for $k = 0$. Consider $k \neq 0$. WLOG, assume $\gcd(N, k) = 1$ and show N is fundamental period. Proof by contradiction.

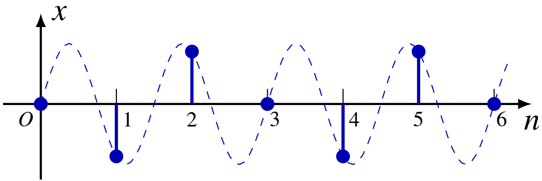
1. Suppose $0 < N_1 < N$ also a period
2. $\omega N_1 = 2\pi m$ for $m \in \mathbb{Z} \setminus \{0\} \implies kN_1 = mN$
3. 2. $\implies N$ divides kN_1
4. $\gcd(N, k) = 1 \implies N$ divides N_1 ↯

Periodicity: Discrete-time Signal

Example. $x[n] = \sin\left(\frac{2\pi}{3}n\right)$ has $N_0 = 3$; its continuous counterpart $x(t) = \sin\left(\frac{2\pi}{3}t\right)$ also has $T_0 = 3$.



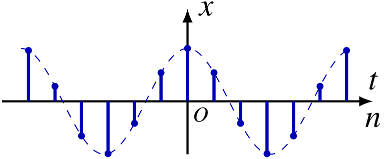
Example. $x[n] = \sin\left(\frac{4\pi}{3}n\right)$ has $N_0 = 3$, but its continuous counterpart $x(t) = \sin\left(\frac{4\pi}{3}t\right)$ has $T_0 = 3/2$ **(!)**.



Even and Odd Signals

Signal is even iff $Rx = x$

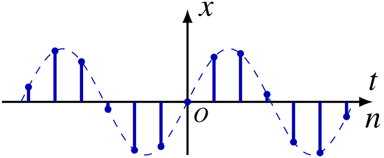
$$x(-t) = x(t) \quad \forall t$$
$$x[-n] = x[n] \quad \forall n$$



Example: $x(t) = \cos t$, $x[n] = \cos n$

Signal is odd iff $Rx = -x$

$$x(-t) = -x(t) \quad \forall t$$
$$x[-n] = -x[n] \quad \forall n$$



Example: $x(t) = \sin t$, $x[n] = \sin n$

Question: What's $x(0)$ if x is odd?

Even-odd Decomposition

Even part

$$\mathcal{E}v(x) = \frac{1}{2}(x + Rx)$$

Odd part

$$\mathcal{O}d(x) = \frac{1}{2}(x - Rx)$$

Even-odd decomposition

$$x = \mathcal{E}v(x) + \mathcal{O}d(x)$$

Check:

- x is even iff $x = \mathcal{E}v(x)$
- x is odd iff $x = \mathcal{O}d(x)$