EI331 Signals and Systems

Lecture 10

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Fourier Series

Fourier series for x with period T and $\omega_0 = \frac{2\pi}{T}$,

$$x(t) = \sum_{k=-\infty}^{\infty} \hat{x}[k] e^{jk\omega_0 t}$$

Correspondence between periodic functions and doubly infinite sequences; time domain vs. frequency domain

$$x \xleftrightarrow{\mathcal{FS}} \hat{x} \qquad \text{or} \qquad x(t) \xleftrightarrow{\mathcal{FS}} \hat{x}[k]$$

- $\hat{x}$ consists of expansion coefficients of x in "basis" $\{e^{jk\omega_0 t}\}$
- $\hat{x}$ alone does **not** uniquely determine x,
- also need to know basis functions, or, equivalently, period T or fundamental frequency ω_0
- same coefficients with different bases (periods) give different functions (more later)

Linearity

If x, y have **same** period T, so does their linear combination $ax + by$, and

$$\widehat{ax + by} = a\hat{x} + b\hat{y}$$

Proof.

$$\begin{aligned}(\widehat{ax + by})[k] &= \langle ax + by, e^{jk\omega_0 t}\rangle \\ &= a\langle x, e^{jk\omega_0 t}\rangle + b\langle y, e^{jk\omega_0 t}\rangle \\ &= a\hat{x}[k] + b\hat{y}[k]\end{aligned}$$

Alternative proof.

$$\left.\begin{aligned} x(t) &= \sum_k \hat{x}[k]e^{jk\omega_0 t} \\ y(t) &= \sum_k \hat{y}[k]e^{jk\omega_0 t}\end{aligned}\right\} \implies ax(t) + by(t) = \sum_k (a\hat{x}[k] + b\hat{y}[k])e^{jk\omega_0 t}$$

Time Shifting

If x has period T and $\omega_0 = \frac{2\pi}{T}$,

$$\widehat{\tau_{t_0} x} = E_{-\omega_0 t_0} \hat{x} \qquad \text{or} \qquad x(t - t_0) \xleftrightarrow{\mathcal{FS}} e^{-jk\omega_0 t_0} \hat{x}[k]$$

where $(E_a \hat{x})[k] = e^{jak} \hat{x}[k]$

Time shift $\iff$ linear phase change in frequency

Proof.

$$\begin{aligned}
(\widehat{\tau_{t_0} x})[k] &= \langle \tau_{t_0} x, e^{jk\omega_0 t} \rangle = \langle x, \tau_{-t_0} e^{jk\omega_0 t} \rangle \\
&= \langle x, e^{jk\omega_0 (t+t_0)} \rangle = \langle x, e^{jk\omega_0 t_0} e^{jk\omega_0 t} \rangle \\
&= e^{-jk\omega_0 t_0} \langle x, e^{jk\omega_0 t} \rangle = e^{-jk\omega_0 t_0} \hat{x}[k] = (E_{-\omega_0 t_0} \hat{x})[k]
\end{aligned}$$

Example. $\cos(t) = \frac{1}{2} e^{jt} + \frac{1}{2} e^{-jt}$, $\sin(t) = \cos(t - \frac{\pi}{2}) = \frac{-j}{2} e^{jt} + \frac{j}{2} e^{-jt}$

Frequency Shifting

If x has period T and $\omega_0 = \frac{2\pi}{T}$,

$$\widehat{E_{m\omega_0} x} = \tau_m \hat{x} \qquad \text{or} \qquad e^{jm\omega_0 t} x(t) \xleftrightarrow{\mathcal{FS}} \hat{x}[k-m]$$

where $(E_a x)(t) = e^{jat} x(t)$

Modulation by harmonic exp. $\Longleftrightarrow$ frequency shift

Proof.

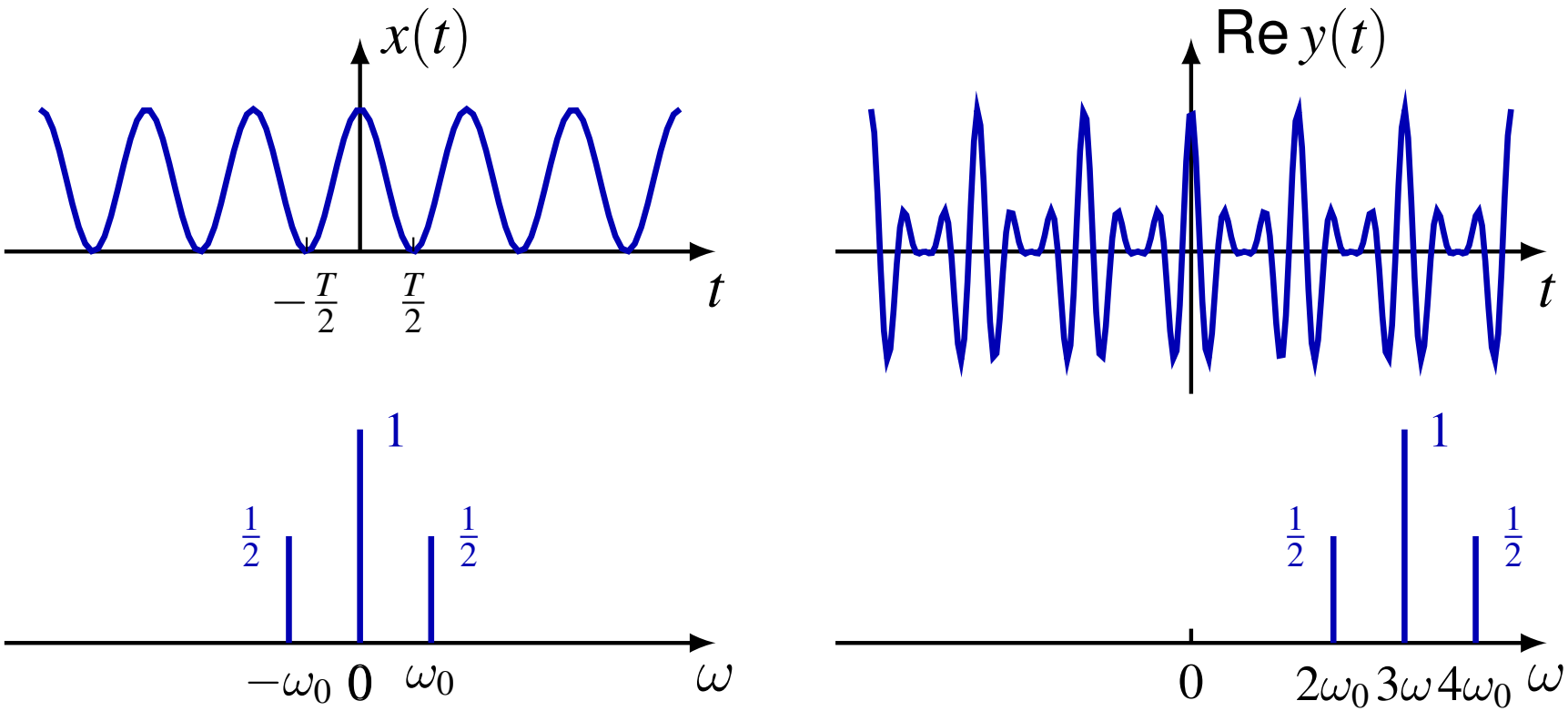
$$\begin{aligned}(\widehat{E_{m\omega_0} x})[k] &= \langle E_{m\omega_0} x, e^{jk\omega_0 t} \rangle = \langle x, E_{-m\omega_0} e^{jk\omega_0 t} \rangle \\ &= \langle x, e^{j(k-m)\omega_0 t} \rangle = \hat{x}[k-m]\end{aligned}$$

NB. Modulation by inharmonic exponential may change fundamental frequency or even result in aperiodic signal

Frequency Shifting

Example. $x(t) = 1 + \cos(\omega_0 t) = 1 + \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$

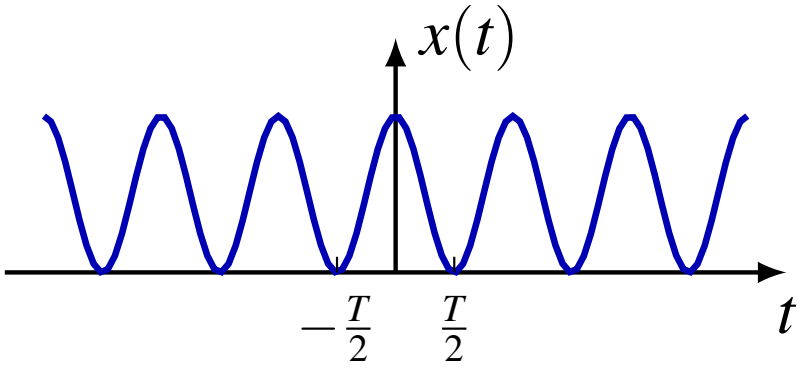
$$y(t) = e^{j3\omega_0 t}x(t) = e^{j3\omega_0 t} + \frac{1}{2}e^{j4\omega_0 t} + \frac{1}{2}e^{j2\omega_0 t}$$

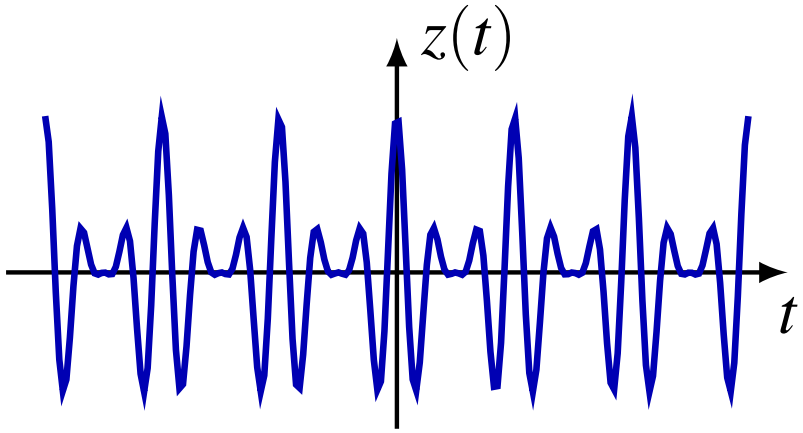


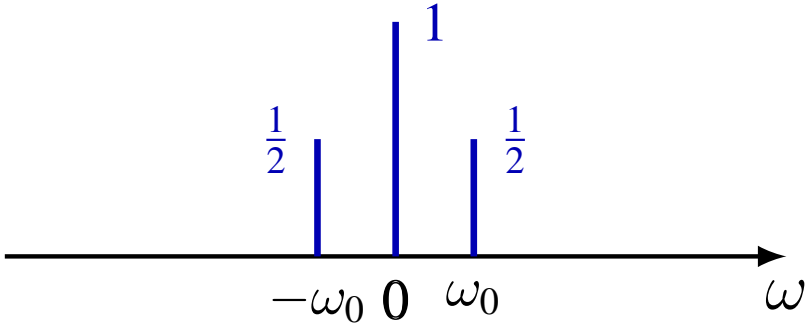
Frequency Shifting

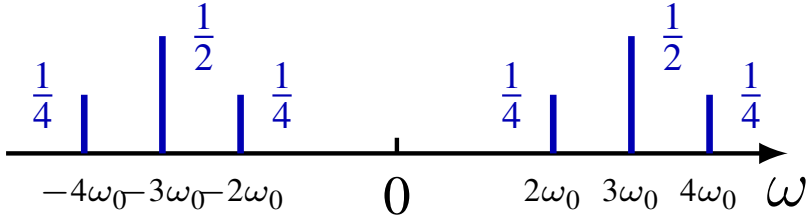
Example. $x(t) = 1 + \cos(\omega_0 t) = 1 + \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$

$$z(t) = \cos(3\omega_0 t)x(t) = \text{Re}\, y(t) = \frac{1}{2}e^{j3\omega_0 t} + \frac{1}{4}e^{j4\omega_0 t} + \frac{1}{4}e^{j2\omega_0 t} + \frac{1}{2}e^{-j3\omega_0 t} + \frac{1}{4}e^{-j4\omega_0 t} + \frac{1}{4}e^{-j2\omega_0 t}$$









Time Reversal

If x has period T,

$$\widehat{Rx} = R\hat{x} \qquad \text{or} \qquad x(-t) \xleftrightarrow{\mathcal{FS}} \hat{x}[-k]$$

Time reversal commutes with Fourier series

$$\begin{tikzcd}
x(t) \arrow[r, "\mathcal{FS}"] \arrow[d, "R"] & \hat{x}[k] \arrow[d, "R"] \\
x(-t) \arrow[r, "\mathcal{FS}"] & \hat{x}[-k]
\end{tikzcd}$$

Proof.

$$(\widehat{Rx})[k] = \langle Rx, e^{jk\omega_0 t}\rangle = \langle x, Re^{jk\omega_0 t}\rangle = \langle x, e^{-jk\omega_0 t}\rangle = \hat{x}[-k]$$

Corollary. Fourier series preserves even/odd symmetry, i.e.

$$x \text{ even} \iff \hat{x} \text{ even}, \quad x \text{ odd} \iff \hat{x} \text{ odd}$$

Time Scaling

If x has period T, then $S_a x$ has period T/a

$$\widehat{S_a x} = \hat{x} \qquad \text{or} \qquad x(at) \xleftrightarrow{\mathcal{FS}} \hat{x}[k]$$

Time scaling preserves Fourier coefficients but changes fundamental frequency

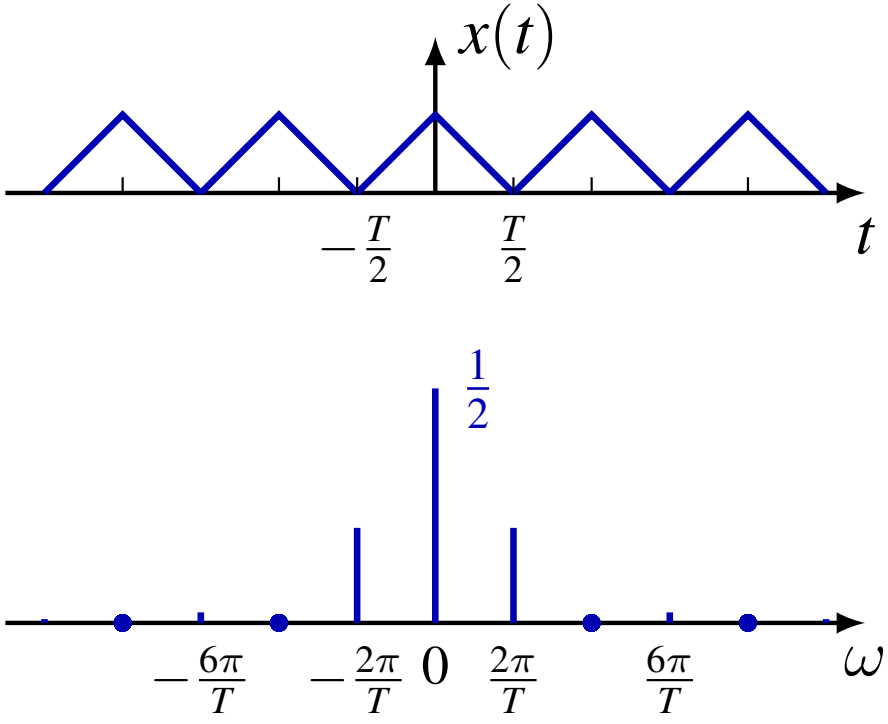
$$x(t) = \sum_{k=-\infty}^{\infty} \hat{x}[k] e^{jk\omega_0 t}$$

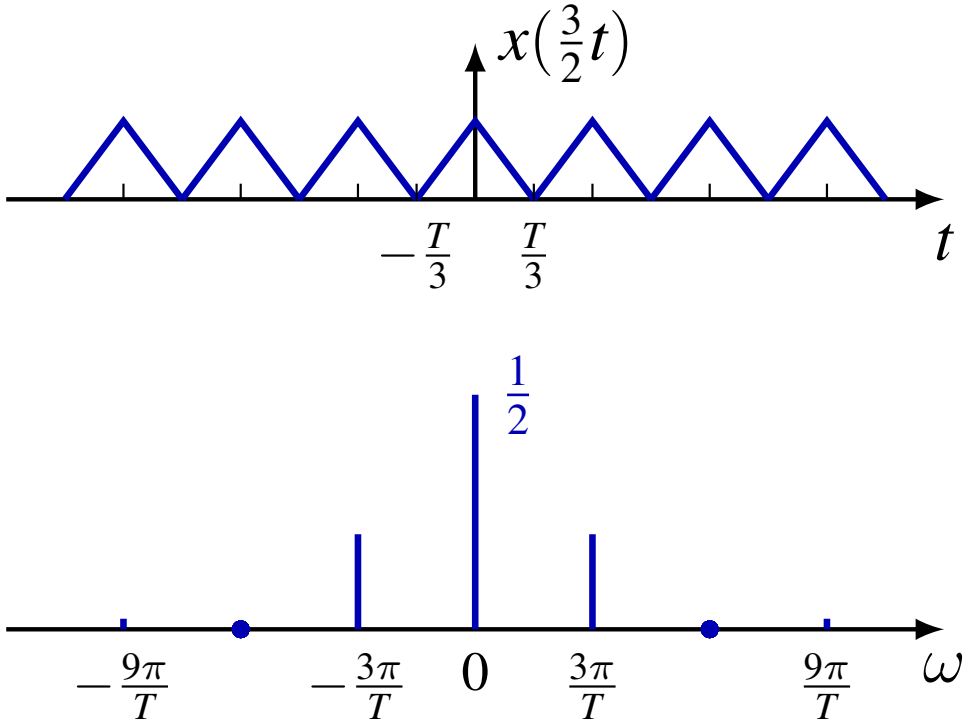
$$x(at) = \sum_{k=-\infty}^{\infty} \hat{x}[k] e^{jk(a\omega_0)t}$$

different Fourier series

Time Scaling

$$x(t) = \sum_{k=-\infty}^{\infty} \hat{x}[k] e^{jk\omega_0 t} \qquad vs. \qquad x(at) = \sum_{k=-\infty}^{\infty} \hat{x}[k] e^{jk(a\omega_0)t}$$





compression in time $\Longleftrightarrow$ expansion in frequency

expansion in time $\Longleftrightarrow$ compression in frequency

Differentiation

If x has period T, so does its derivative x', and

$$\widehat{x'} = M_{\omega_0}\hat{x} \qquad \text{or} \qquad x'(t) \xleftrightarrow{\mathcal{FS}} jk\omega_0\hat{x}[k]$$

where $(M_a\hat{x})[k] = jak\hat{x}[k]$

differentiation in time $\iff$ multiplication by $jk\omega_0$ in frequency

Proof.

$$\begin{aligned}\widehat{x'}[k] &= \frac{1}{T}\int_T x'(t)e^{-jk\omega_0 t}dt \\ &= \frac{1}{T}[x(t)e^{-jk\omega_0 t}]\Big|_0^T + jk\omega_0\left(\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt\right) = jk\omega_0\hat{x}[k]\end{aligned}$$

Alternatively, differentiate term by term

$$x(t) = \sum_k \hat{x}[k]e^{jk\omega_0 t} \implies x'(t) = \sum_k jk\omega_0\hat{x}[k]e^{-jk\omega_0 t}$$

Integration

If x has period T and Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} \hat{x}[k] e^{jk\omega_0 t}$$

Integrating term by term

$$y(t) \triangleq \int_0^t x(s)ds = \hat{x}[0]t + \sum_{k\neq 0} \hat{x}[k] \frac{e^{jk\omega_0 t} - 1}{jk\omega_0}$$

$$= \hat{x}[0]t - \sum_{k\neq 0} \frac{\hat{x}[k]}{jk\omega_0} + \sum_{k\neq 0} \frac{1}{jk\omega_0} \hat{x}[k] e^{jk\omega_0 t}$$

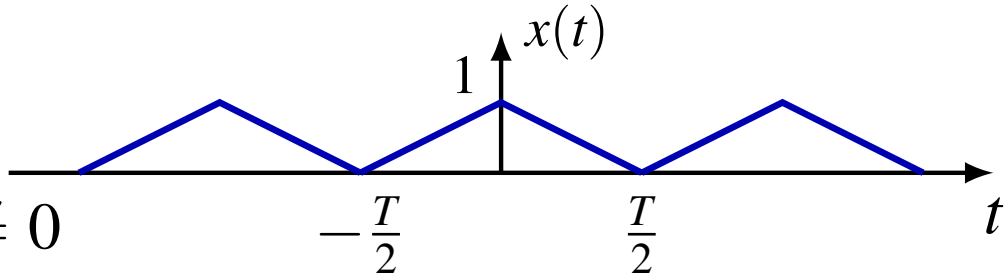
- y periodic iff $\hat{x}[0] = 0$, i.e. x has no DC component
- if $\hat{x}[0] = 0$, y has period T,

$$\hat{y}[k] = \frac{1}{jk\omega_0} \hat{x}[k] \quad \text{for } k \neq 0$$

Example: Triangle Wave

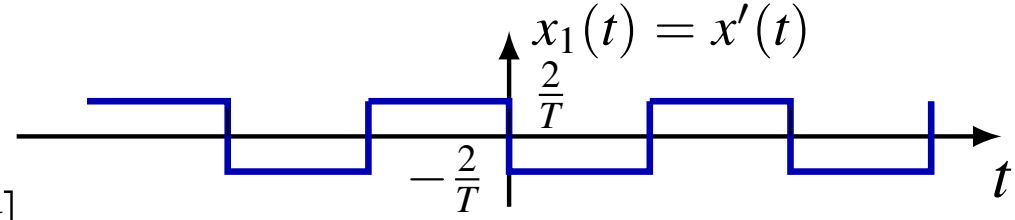
$$x' = x_1$$

$$\hat{x}[k] = \frac{1}{jk\omega_0}\frac{4\sin(k\pi/2)}{k\pi T}e^{jk\pi/2}, k \neq 0$$



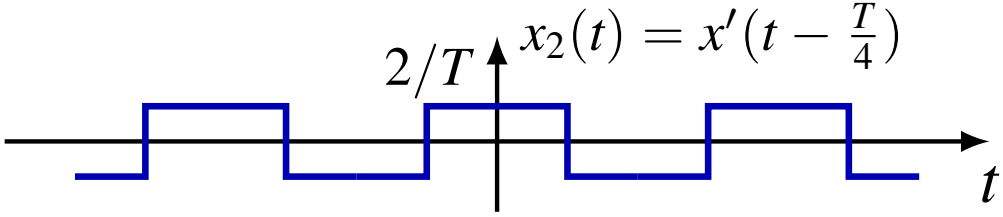
$$x_1 = \tau_{-T/4}x_2$$

$$\hat{x}_1[k] = \frac{4\sin(k\pi/2)}{k\pi T}e^{jk\pi/2} - \frac{2}{T}\delta[k]$$

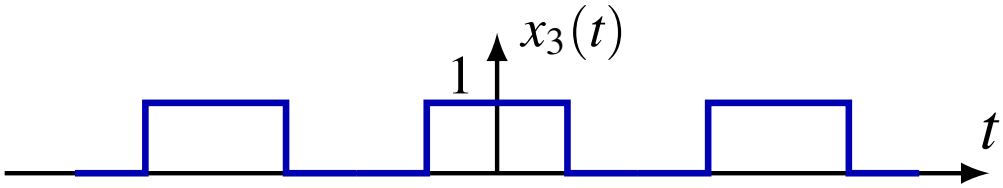


$$x_2 = \frac{4}{T}x_3 - \frac{2}{T}$$

$$\hat{x}_2[k] = \frac{4\sin(k\pi/2)}{k\pi T} - \frac{2}{T}\delta[k]$$



$$\hat{x}_3[k] = \frac{\sin(k\pi/2)}{k\pi}$$



Multiplication

If x and y have **same** period T, so does their product xy, and

$$\widehat{xy} = \hat{x} * \hat{y} \qquad \text{or} \qquad x(t)y(t) \xleftrightarrow{\mathcal{FS}} \sum_{m=-\infty}^{\infty} \hat{x}[m]\hat{y}[k-m]$$

multiplication in time $\Longleftrightarrow$ convolution in frequency

Proof.

$$\begin{aligned} x(t)y(t) &= \left(\sum_{m=-\infty}^{\infty} \hat{x}[m]e^{jm\omega_0 t}\right)\left(\sum_{\ell=-\infty}^{\infty} \hat{y}[\ell]e^{j\ell\omega_0 t}\right) \\ &= \sum_{m=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} \hat{x}[m]\hat{y}[\ell]e^{j(m+\ell)\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty}\left(\sum_{m=-\infty}^{\infty} \hat{x}[m]\hat{y}[k-m]\right)e^{jk\omega_0 t} \qquad (k = m+\ell) \end{aligned}$$

Periodic Convolution

Periodic convolution $x * y$ of x and y with **same** period T

$$(x * y)(t) = \int_T x(\tau)y(t-\tau)d\tau$$

Properties

- Commutativity

$$x * y = y * x$$

- Associativity

$$(x * y) * z = x * (y * z)$$

- Bilinearity

$$\left(\sum_i a_i x_i\right) * \left(\sum_j b_j y_j\right) = \sum_{i,j} a_i b_j (x_i * y_j)$$

Periodic Convolution

Fourier coefficients satisfy

$$\widehat{x * y} = T\hat{x}\hat{y} \qquad \text{or} \qquad (x * y)(t) \xleftrightarrow{\mathcal{FS}} T\hat{x}[k]\hat{y}[k]$$

convolution in time $\Longleftrightarrow$ multiplication in frequency

Proof.

$$\begin{aligned}
(\widehat{x * y})[k] &= \frac{1}{T}\int_T (x * y)(t) e^{-jk\omega_0 t} dt \\
&= \frac{1}{T}\int_T \left(\int_T x(\tau) y(t-\tau) d\tau\right) e^{-jk\omega_0 t} dt \\
&= \int_T x(\tau) \left(\frac{1}{T}\int_T y(t-\tau) e^{-jk\omega_0 t} dt\right) d\tau \\
&= \int_T x(\tau) \left(e^{-jk\omega_0 \tau}\hat{y}[k]\right) d\tau = T\hat{x}[k]\hat{y}[k]
\end{aligned}$$

Conjugation and Symmetry

If x has period T, so does its complex conjugate $\bar{x} = x^*$, and

$$\widehat{x^*} = R\hat{x}^* \qquad \text{or} \qquad x^*(t) \xleftrightarrow{\mathcal{FS}} (\hat{x}[-k])^*$$

Proof.

$$\widehat{x^*}[k] = \langle x^*, e^{jk\omega_0 t} \rangle = \overline{\langle x, e^{-jk\omega_0 t} \rangle} = \overline{\hat{x}[-k]}$$

Corollary. If x is real, $\hat{x}$ is conjugate symmetric i.e.

$$\hat{x}[-k] = \overline{\hat{x}[k]}$$

Corollary. If x is real and even, $\hat{x}$ is also real and even

$$\hat{x}[k] = \hat{x}[-k] = \overline{\hat{x}[k]}$$

Corollary. If x is real and odd, $\hat{x}$ is purely imaginary and odd

$$-\hat{x}[k] = \hat{x}[-k] = \overline{\hat{x}[k]}$$

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Vector Space

Vector space V over field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) has two operations

- addition: $x, y \in V \implies x + y \in V$
- scalar multiplication: $\lambda \in \mathbb{F},\ x \in V \implies \lambda x \in V$

satisfying following axioms

1. commutativity of addition: $x + y = y + x$, $\forall x, y \in V$
2. associativity of addition: $(x + y) + z = x + (y + z)$, $\forall x, y, z \in V$
3. identity element of addition: $\exists 0 \in V$ s.t. $x + 0 = x$, $\forall x \in V$
4. inverse elements of addition: $\forall x \in V, \exists y \in V$ s.t. $x + y = 0$
5. identity element of scalar multiplication: $1x = x$, $\forall x \in V$
6. compatibility of scalar and field multiplications
   $\lambda_1(\lambda_2 x) = (\lambda_1 \lambda_2)x$, $\forall \lambda_1, \lambda_2 \in \mathbb{F}, x \in V$
7. distributivity: $\lambda(x + y) = \lambda x + \lambda y$, $\forall \lambda \in \mathbb{F}, x, y \in V$
8. distributivity: $(\lambda_1 + \lambda_2)x = \lambda_1 x + \lambda_2 x$, $\forall \lambda_1, \lambda_2 \in \mathbb{F}, x \in V$

Vector Space

Example. $\mathbb{R}^n$, $\mathbb{C}^n$

$$(x_1, \ldots, x_n)^T + (y_1, \ldots, y_n)^T = (x_1 + y_1, \ldots, x_n + y_n)^T$$

$$\lambda(x_1, \ldots, x_n)^T = (\lambda x_1, \ldots, \lambda x_n)^T$$

Example. CT real-/complex-valued signals $\mathbb{R}^{\mathbb{R}}$, $\mathbb{C}^{\mathbb{R}}$

$$(x + y)(t) = x(t) + y(t)$$

$$(\lambda x)(t) = \lambda x(t)$$

Example. DT real-/complex-valued signals $\mathbb{R}^{\mathbb{Z}}$, $\mathbb{C}^{\mathbb{Z}}$

$$(x + y)[n] = x[n] + y[n]$$

$$(\lambda x)[n] = \lambda x[n]$$

Normed Vector Space

Norm on vector space V over field $\mathbb{F}$

$$\|\cdot\| : V \to \mathbb{R}_+$$

satisfies

- Positive definiteness

$$\|x\| = 0 \iff x = 0$$

- Absolute homogeneity

$$\|\lambda x\| = |\lambda| \cdot \|x\|, \quad \forall \lambda \in \mathbb{F}, x \in V$$

- Triangle inequality

$$\|x + y\| \leq \|x\| + \|y\|$$

Vector space with norm called normed vector space

Normed Vector Space

Example. $\mathbb{R}^n$, $\mathbb{C}^n$

$$\|x\|_p = \begin{cases} \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}, & 1 \le p < \infty \\ \max_{1\le k\le n} |x_k|, & p = \infty \end{cases}$$

Example. CT real-/complex-valued signals $\mathbb{R}^{\mathbb{R}}$, $\mathbb{C}^{\mathbb{R}}$

$$\|x\|_p = \begin{cases} \left(\int |x(t)|^p dt\right)^{1/p}, & 1 \le p < \infty \\ \sup_t |x(t)|, & p = \infty \end{cases}$$

Example. DT real-/complex-valued signals $\mathbb{R}^{\mathbb{Z}}$, $\mathbb{C}^{\mathbb{Z}}$

$$\|x\|_p = \begin{cases} \left(\sum_{k=-\infty}^{\infty} |x[k]|^p\right)^{1/p}, & 1 \le p < \infty \\ \sup_{k\in\mathbb{Z}} |x[k]|, & p = \infty \end{cases}$$

Inner Product Space

Inner product on vector space V over field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$)
$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ satisfying

- Conjugate symmetry

$$\langle x, y \rangle = \overline{\langle y, x \rangle}$$

- Linearity in first argument

$$\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle x, z \rangle$$

- Positive definiteness

$$\langle x, x \rangle \geq 0$$

$$\langle x, x \rangle = 0 \iff x = 0$$

Vector space with inner product called inner product space

Inner Product Space

Example. $\mathbb{R}^n$, $\mathbb{C}^n$

$$\langle x, y \rangle = \sum_{k=1}^{n} x_k \bar{y}_k$$

Example. CT real-/complex-valued signals $\mathbb{R}^{\mathbb{R}}$, $\mathbb{C}^{\mathbb{R}}$

$$\langle x, y \rangle = \int_{\mathbb{R}} x(t)\overline{y(t)}dt$$

Example. DT real-/complex-valued signals $\mathbb{R}^{\mathbb{Z}}$, $\mathbb{C}^{\mathbb{Z}}$

$$\langle x, y \rangle = \sum_{k=-\infty}^{\infty} x[k]\overline{y[k]}$$

In all cases

$$\|x\|_2 = \sqrt{\langle x, x \rangle}$$

Will write $\|\cdot\|$ instead of $\|\cdot\|_2$ for inner product spaces

Cauchy-Schwarz Inequality

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$$

Proof. If $\|y\| = 0$, then $y = 0$ and $\langle x, y\rangle = \|x\| \cdot \|y\|$. Now assume $\|y\| \neq 0$. For any $\lambda \in \mathbb{C}$,

$$0 \leq \langle x - \lambda y, x - \lambda y\rangle = \|x\|^2 - \lambda\overline{\langle x, y\rangle} - \bar{\lambda}\langle x, y\rangle + |\lambda|^2\|y\|^2$$

Let $\lambda = \langle x, y\rangle / \|y\|^2$,

$$0 \leq \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2} - \frac{|\langle x, y\rangle|^2}{\|y\|^2} + \frac{|\langle x, y\rangle|^2}{\|y\|^2} \implies |\langle x, y\rangle| \leq \|x\| \cdot \|y\|$$

Integral form

$$\left|\int_{\mathbb{R}} x(t)\overline{y(t)}dt\right| \leq \sqrt{\int_{\mathbb{R}} |x(t)|^2 dt} \cdot \sqrt{\int_{\mathbb{R}} |y(t)|^2 dt}$$

Orthonormal System in Inner Product Space

Two vectors $x, y \in V$ are orthogonal if $\langle x, y \rangle = 0$

Vector $x \in V$ is unit vector if $\|x\| = 1$ or $\langle x, x \rangle = 1$

$\{x_k\}$ is orthogonal system if $\langle x_k, x_m \rangle = 0$ for $m \neq k$

$\{e_k\}$ is orthonormal system if $\langle e_k, e_m \rangle = \delta_{km} = \delta[k - m]$

Example. Let V be set of periodic functions with period T. Inner product

$$\langle x, y \rangle = \frac{1}{T} \int_T x(t)\overline{y(t)}dt$$

Let $e_k(t) = e^{jk\frac{2\pi}{T}t}$. Recall $\{e_k : k \in \mathbb{Z}\}$ is orthonormal system

$$\langle e_k, e_m \rangle = \delta[k - m]$$

Orthonormal Expansion

Given orthonormal system $\{e_k\}$, suppose

$$x = \sum_k c_k e_k$$

Find coefficients by taking inner product with e_m

$$\langle x, e_m \rangle = \langle \sum_k c_k e_k, e_m \rangle = \sum_k c_k \langle e_k, e_m \rangle = \sum_k c_k \delta[k-m] = c_m$$

Generalized Fourier series

$$x = \sum_k \langle x, e_k \rangle e_k$$

This is how we do Fourier series expansion

Best Mean-square Approximation

Mean-square error in approximation of x by element in span$\{e_1, \ldots, e_n\}$, i.e. by element of form $y = \sum_{k=1}^{n} a_k e_k$

$$
\begin{aligned}
\|x - y\|^2 &= \langle x - \sum_{k=1}^{n} a_k e_k, x - \sum_{k=1}^{n} a_k e_k \rangle \\
&= \|x\|^2 - \sum_{k=1}^{n} a_k \langle e_k, x \rangle - \sum_{k=1}^{n} \bar{a}_k \langle x, e_k \rangle + \sum_{k=1}^{n} |a_k|^2 \\
&= \|x\|^2 - 2 \sum_{k=1}^{n} \mathrm{Re}\,(\bar{a}_k \langle x, e_k \rangle) + \sum_{k=1}^{n} |a_k|^2 \\
&\geq \|x\|^2 - 2 \sum_{k=1}^{n} |a_k| \cdot |\langle x, e_k, \rangle| + \sum_{k=1}^{n} |a_k|^2 \quad (\text{use } |\mathrm{Re}\, z| \leq |z|) \\
&= \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 + \sum_{k=1}^{n} (|a_k| - |\langle x, e_k, \rangle|)^2
\end{aligned}
$$

Best Mean-square Approximation

Mean-square error in approximation

$$\|x-y\|^2 \overset{(1)}{\geq} \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k\rangle|^2 + \sum_{k=1}^{n} (|a_k| - |\langle x, e_k, \rangle|)^2$$

$$\overset{(2)}{\geq} \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k\rangle|^2$$

Minimum mean-square error achieved iff $a_k = \langle x, e_k\rangle$

- equality in (1) requires $\arg a_k = \arg\langle x, e_k\rangle$
- equality in (2) requires $|a_k| = |\langle x, e_k\rangle|$

Fourier partial sum $S_N(x) = \sum_{k=-N}^{N} \hat{x}[k] e^{jk\omega_0 t}$ is best mean-square approximation

$$\|x - S_N(x)\|^2 = \min_{a_k, -N\leq k\leq N} \|x - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}\|^2$$

Bessel's Inequality

Minimum mean-square error

$$\|x - \sum_{k=1}^{n} \langle x, e_k \rangle e_k\|^2 = \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k \rangle|^2$$

Bessel's inequality

$$\sum_{k} |\langle x, e_k \rangle|^2 \leq \|x\|^2$$

For Fourier coefficients of signals with finite 2-norm

$$\sum_{k=-\infty}^{\infty} |\hat{x}[k]|^2 \leq \|x\|^2$$

Corollary. $\lim_{k \to \infty} \hat{x}[k] = 0$

Parseval's Identity

For Fourier coefficients of signals with finite 2-norm

$$\sum_{k=-\infty}^{\infty} |\hat{x}[k]|^2 = \|x\|^2 = \frac{1}{T}\int_T |x(t)|^2 dt$$

Interpretation: Energy conservation

- $|\hat{x}[k]|^2$ is average power of k-th harmonic component
- $\|x\|^2$ is average power of x
- total power is sum of powers of all components

Theorem. Fourier series converges in mean-square

$$\lim_{N\to\infty} \|x - S_N(x)\| = 0$$

NB. Convergence in mean-square does **not** imply pointwise convergence