EI331 Signals and Systems

Lecture 12

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April 4, 2019

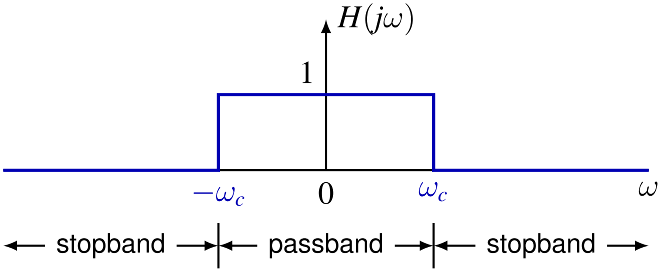
Contents

Ideal Frequency-selective Filters

Ideal lowpass filter

$$H(j\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \text{otherwise} \end{cases}$$

ω_c: cutoff frequency

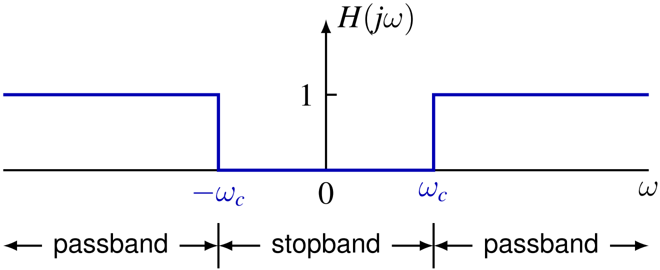


Ideal Frequency-selective Filters

Ideal highpass filter

$$H(j\omega) = \begin{cases} 1, & |\omega| \geq \omega_c \\ 0, & \text{otherwise} \end{cases}$$

ω_c: cutoff frequency

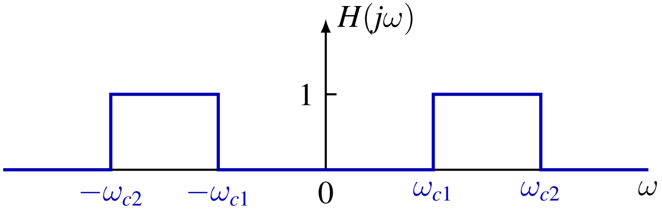


Ideal Frequency-selective Filters

Ideal bandpass filter

$$H(j\omega) = \begin{cases} 1, & \omega_{c1} \le |\omega| \le \omega_{c2} \\ 0, & \text{otherwise} \end{cases}$$

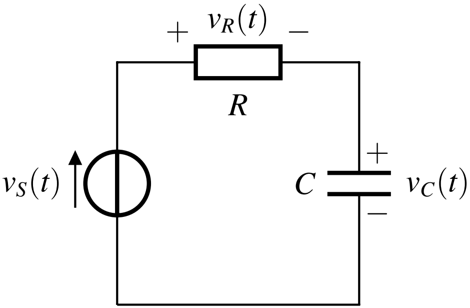
ω_{c1}: lower cutoff frequency
ω_{c2}: upper cutoff frequency



Simple RC Lowpass Filter

ODE

$$RC\frac{dv_C(t)}{dt} + v_C(t) = v_S(t)$$



For input

$$v_S(t) = e^{j\omega t}$$

output

$$v_C(t) = H(j\omega)e^{j\omega t}$$

Frequency response $\quad H(j\omega) = \dfrac{1}{1 + RCj\omega}$

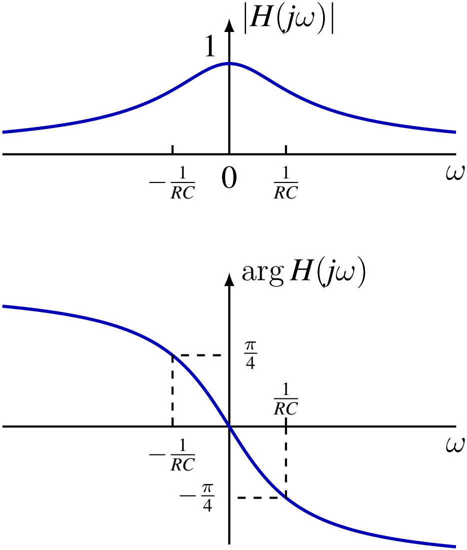
Simple RC Lowpass Filter

Frequency response

$$H(j\omega) = \frac{1}{1 + RCj\omega}$$

$$|H(j\omega)| = \frac{1}{\sqrt{1 + (RC\omega)^2}}$$

$$\arg H(j\omega) = -\arctan(RC\omega)$$



Nonideal lowpass filter

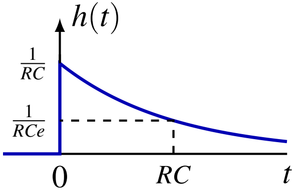
passes lower frequencies

attenuates higher frequencies

Larger $RC \implies$ passes smaller range of lower frequencies

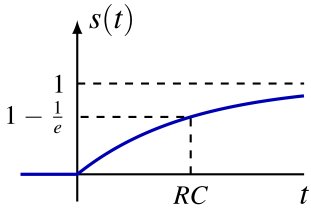
Simple RC Lowpass Filter

Impulse response



$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

Step response



$$s(t) = (h * u)(t) = (1 - e^{-t/RC}) u(t)$$

Time constant $\tau = RC$

- larger τ, more sluggish response

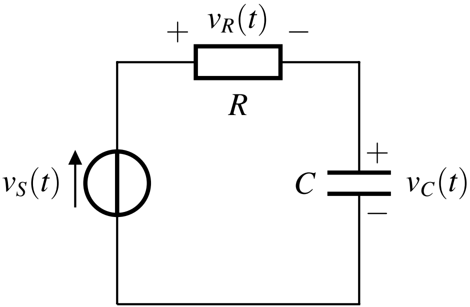
Tradeoff

- larger τ, passes fewer higher frequencies, more sluggish response
- smaller τ, passes more higher frequencies, faster response

Simple RC Highpass Filter

ODE

$$RC\frac{dv_R(t)}{dt} + v_R(t) = RC\frac{dv_S(t)}{dt}$$



For input

$$v_S(t) = e^{j\omega t}$$

output

$$v_R(t) = H(j\omega)e^{j\omega t}$$

Frequency response $\quad H(j\omega) = \dfrac{j\omega RC}{1 + j\omega RC}$

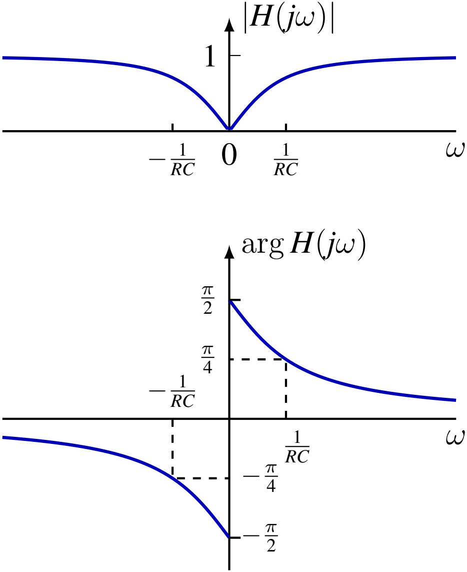
Simple RC Highpass Filter

Frequency response

$$H(j\omega) = \frac{j\omega RC}{1 + j\omega RC}$$

$$|H(j\omega)| = \frac{|\omega| RC}{\sqrt{1 + (RC\omega)^2}}$$

$$\arg H(j\omega) = \arctan \frac{1}{RC\omega}$$



Nonideal highpass filter

passes higher frequencies

attenuates lower frequencies

Larger $RC \implies$ passes larger range of lower frequencies

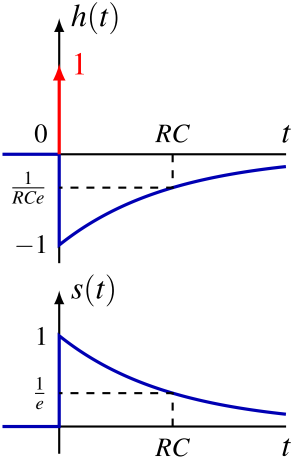
Simple RC Highpass Filter

Step response

$$s(t) = e^{-t/RC} u(t)$$

Impulse response

$$h(t) = s'(t) = \delta(t) - \frac{1}{RC} e^{-t/RC} u(t)$$



Time constant $\tau = RC$

- larger τ, more sluggish response

Observations

- larger τ, passes more lower frequencies, more sluggish response
- smaller τ, passes fewer lower frequencies, faster response

Contents

DT Periodic Signals

Recall DT signal is periodic with period N if

$$x = \tau_N x \quad \text{or} \quad x[n] = x[n+N], \forall n \in \mathbb{Z}$$

- fundamental period N: smallest positive period
- fundamental frequency $\begin{cases} \frac{2\pi}{N}, & \text{if } N > 1 \\ 0, & \text{if } N = 1 \end{cases}$

Complex exponential $\phi_N^k[n] = e^{jk\frac{2\pi}{N}n} = e^{jk\omega_0 n}$ is periodic with

- period N and fundamental period $\frac{N}{\gcd(N,k)}$
- fundamental frequency

$$\omega_k = \begin{cases} 0, & \text{if } N \mid k \\ \omega_0 \cdot \gcd(k,N), & \text{otherwise} \end{cases}$$

always integer multiple of $\omega_0 = \frac{2\pi}{N}$

Finiteness of DT Fourier Basis

Fourier series represent N-periodic signals in terms of harmonically related complex exponentials ϕ_N^k

$$x = \sum_k c_k \phi_N^k, \quad \text{or} \quad x[n] = \sum_k c_k \phi_N^k[n] = \sum_k c_k e^{jk\frac{2\pi}{N}n}$$

Key difference with CT case

$\phi_N^{k+rN} = \phi_N^k$, so only N **distinct** ϕ_N^k, Fourier basis is **finite**, i.e.

$$\{\phi_N^k : k \in \mathbb{Z}\} = \{\phi_N^k : k \in [N]\}$$

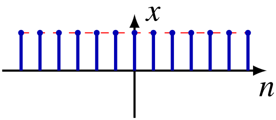
where $[N] = \{0, 1, \ldots, N-1\}$ (can think $[N] = \{\bar{0}, \bar{1}, \ldots, \overline{N-1}\}$)

Proof. For $r \in \mathbb{Z}$,

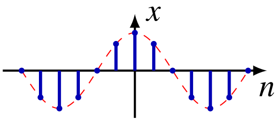
$$\phi_N^{k+rN}[n] = e^{j(k+rN)\frac{2\pi}{N}n} = e^{jk\frac{2\pi}{N}n} e^{jr2\pi n} = e^{jk\frac{2\pi}{N}n} = \phi_N^k[n]$$

Finiteness of DT Fourier Basis

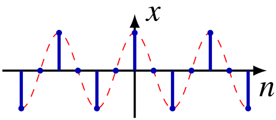
ϕ_N^k for $N = 4$ and $k = 0, 1, \ldots, 8$



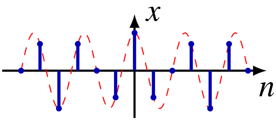
$\phi_N^0[n] = \cos(0 \cdot n) = 1$



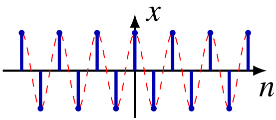
$\phi_N^1[n] = \cos(\pi n/4)$



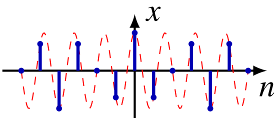
$\phi_N^2[n] = \cos(\pi n/2)$



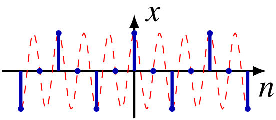
$\phi_N^3[n] = \cos(3\pi n/4)$



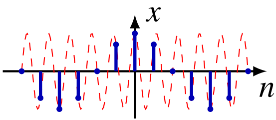
$\phi_N^4[n] = \cos(\pi n)$



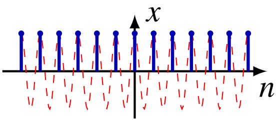
$\phi_N^5[n] = \cos(5\pi n/4)$



$\phi_N^6[n] = \cos(3\pi n/2)$



$\phi_N^7[n] = \cos(7\pi n/4)$



$\phi_N^8[n] = \cos(2\pi n) = 1$

Orthonormality of Harmonics

DT Fourier series

$$x = \sum_{k \in [N]} \hat{x}[k] \phi_N^k, \qquad \text{or} \qquad x[n] = \sum_{k \in [N]} \hat{x}[k] e^{jk\frac{2\pi}{N}n}$$

Summation can also be taken over **any** N successive integers.

Find coefficients $\hat{x}[k]$ using orthonormality of harmonics.

Define inner product between two signals with period N by

$$\langle x, y \rangle = \frac{1}{N} \sum_{n \in [N]} x[n]\overline{y[n]}$$

Same as inner product in $\mathbb{C}^N$ up to factor N^{-1}

$\{\phi_N^k : k \in [N]\}$ is orthonormal system of functions

$$\langle \phi_N^k, \phi_N^m \rangle = \delta_{km} = \delta[k - m]$$

Proof of Orthonormality of Harmonics

$\{\phi_N^k : k \in [N]\}$ is orthonormal system of functions, i.e.

$$\langle \phi_N^k, \phi_N^m \rangle = \delta_{km} = \delta[k-m]$$

Proof.
$$\langle \phi_N^k, \phi_N^m \rangle = \frac{1}{N} \sum_{n \in [N]} e^{jk\frac{2\pi}{N}n} e^{-jm\frac{2\pi}{N}n} = \frac{1}{N} \sum_{n=0}^{N-1} e^{j(k-m)\frac{2\pi}{N}n}$$

If $k = m$,
$$\langle \phi_N^k, \phi_N^m \rangle = \frac{1}{N} \sum_{k=0}^{N-1} 1 = 1$$

If $k \neq m$, since $|k-m| \leq N-1$, $e^{j(k-m)\frac{2\pi}{N}} \neq 1$. By $\sum_{n=n_1}^{n_2} a^n = \frac{a^{n_1} - a^{n_2+1}}{1-a}$,

$$\langle \phi_N^k, \phi_N^m \rangle = \frac{1}{N} \sum_{n=0}^{N-1} e^{j(k-m)\frac{2\pi}{N}n} = \frac{1}{N} \frac{1 - e^{j(k-m)\frac{2\pi}{N}N}}{1 - e^{j(k-m)\frac{2\pi}{N}}} = 0$$

Fourier Coefficients

Suppose x has period N and Fourier series representation

$$x = \sum_{k \in [N]} \hat{x}[k] \phi_N^k$$

For $m \in [N]$,

$$\begin{aligned} \langle x, \phi_N^m \rangle &= \langle \sum_{k \in [N]} \hat{x}[k] \phi_N^k, \phi_N^m \rangle = \sum_{k \in [N]} \hat{x}[k] \langle \phi_N^k, \phi_N^m \rangle \\ &= \sum_{k \in [N]} \hat{x}[k] \delta[m-k] = \hat{x}[m] \end{aligned}$$

Can thick of $\hat{x}$ as N-periodic signal, since

$$\hat{x}[m + rN] \triangleq \langle x, \phi_N^{m+rN} \rangle = \langle x, \phi_N^m \rangle = \hat{x}[m]$$

but use **only N successive values** in Fourier series!

DT Fourier Series

Synthesis equation

$$x[n] = \sum_{k \in [N]} \hat{x}[k] \phi_N^k[n] = \sum_{k \in [N]} \hat{x}[k] e^{jk\frac{2\pi}{N}n}$$

Analysis equation

$$\hat{x}[k] = \langle x, \phi_N^k \rangle = \frac{1}{N} \sum_{n \in [N]} x[n] e^{-jk\frac{2\pi}{N}n}$$

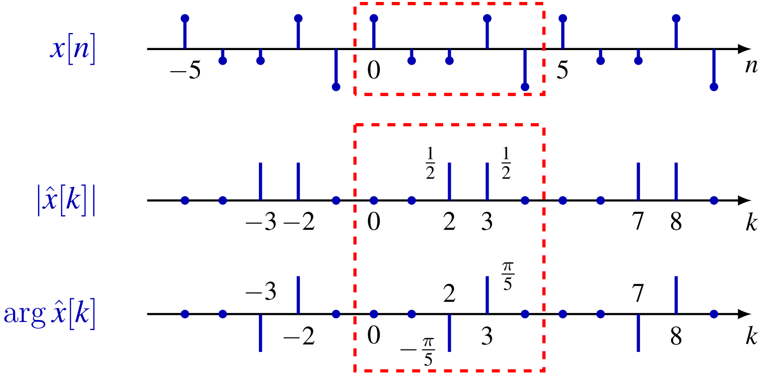
No convergence issue since all sums are finite!

Example

$x[n] = \cos(\frac{6\pi}{5}n + \frac{\pi}{5}) = \frac{e^{j\frac{\pi}{5}}}{2} e^{j3\frac{2\pi}{5}n} + \frac{e^{-j\frac{\pi}{5}}}{2} e^{-j3\frac{2\pi}{5}n}$, period $N = 5$

$$\hat{x}[3] = \frac{1}{2} e^{j\frac{\pi}{5}}; \quad \hat{x}[2] = \hat{x}[-3] = \frac{1}{2} e^{-j\frac{\pi}{5}}; \quad \hat{x}[0] = \hat{x}[1] = \hat{x}[4] = 0$$

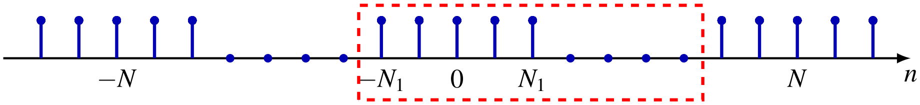
$\hat{x}[k]$ repeats with period $N = 5$



Example: Periodic Square Wave

Periodic square wave with period N, in one period

$$x[n] = \begin{cases} 1, & -N_1 \le n \le N_1 \\ 0, & N_1 < n < N - N_1 \end{cases}$$



Fourier coefficients

$$\hat{x}[k] = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk\frac{2\pi}{N}n}$$

If k is integer multiple of N, $\hat{x}[k] = \frac{2N_1+1}{N}$

Example: Periodic Square Wave

If k is not integer multiple of N, then $e^{-jk\frac{2\pi}{N}} \neq 1$. Using

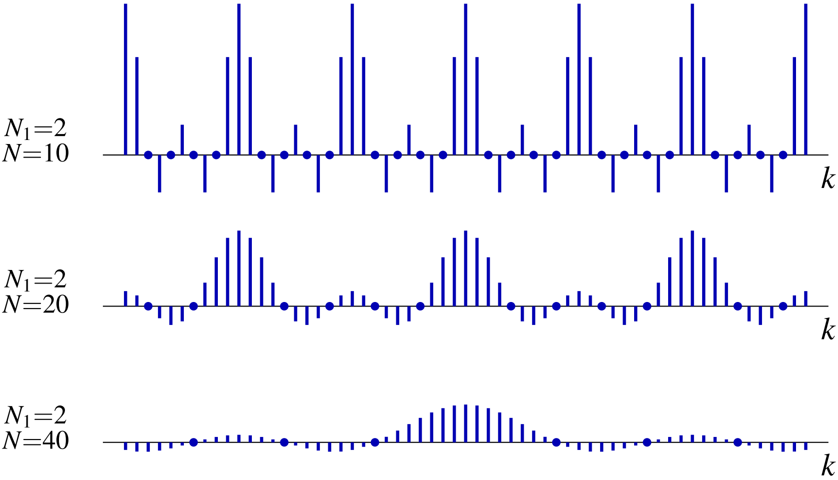
$$\sum_{n=m}^{M} a^n = \frac{a^m - a^{M+1}}{1-a}$$

we obtain

$$\begin{aligned}
\hat{x}[k] &= \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk\frac{2\pi}{N}n} = \frac{1}{N} \frac{e^{jk\frac{2\pi}{N}N_1} - e^{-jk\frac{2\pi}{N}(N_1+1)}}{1 - e^{-jk\frac{2\pi}{N}}} \\
&= \frac{1}{N} \frac{e^{jk\frac{2\pi}{N}(N_1+\frac{1}{2})} - e^{-jk\frac{2\pi}{N}(N_1+\frac{1}{2})}}{e^{jk\frac{\pi}{N}} - e^{-jk\frac{\pi}{N}}} \\
&= \frac{1}{N} \frac{\sin(k\frac{2\pi}{N}(N_1 + \frac{1}{2}))}{\sin(k\frac{\pi}{N})}
\end{aligned}$$

Example: Periodic Square Wave

$$\hat{x}[k] = \frac{1}{N}\frac{\sin(k\frac{2\pi}{N}(N_1+\frac{1}{2}))}{\sin(k\frac{\pi}{N})}$$



DT Fourier Series: Matrix Form

Synthesis equation

$$x[n] = \sum_{k\in[N]} \hat{x}[k]e^{jk\frac{2\pi}{N}n} = \sum_{k=0}^{N-1} W_N^{kn}\hat{x}[k], \quad \text{where } W_N = e^{jk\frac{2\pi}{N}}$$

$$\begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-2] \\ x[N-1] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W_N & W_N^2 & \dots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \dots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-2} & W_N^{2(N-2)} & \dots & W_N^{(N-1)(N-2)} \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \dots & W_N^{(N-1)^2} \end{pmatrix} \begin{pmatrix} \hat{x}[0] \\ \hat{x}[1] \\ \hat{x}[2] \\ \vdots \\ \hat{x}[N-2] \\ \hat{x}[N-1] \end{pmatrix}$$

$$\mathbf{F} = (\phi_N^0, \phi_N^1, \dots, \phi_N^{N-1})$$

DT Fourier Series: Matrix Form

Synthesis equation

$$x[n] = \sum_{k \in [N]} \hat{x}[k] e^{jk\frac{2\pi}{N}n} = \sum_{k=0}^{N-1} W_N^{kn} \hat{x}[k], \quad \text{where } W_N = e^{jk\frac{2\pi}{N}}$$

Matrix form

$$x = \mathbf{F}\hat{x}, \quad \text{where } \mathbf{F} = (\phi_N^0, \phi_N^1, \ldots, \phi_N^{N-1})$$

Analysis equation

$$\hat{x}[k] = \frac{1}{N} \sum_{n \in [N]} x[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N} \sum_{n=0}^{N-1} \bar{W}_N^{kn} x[n]$$

Matrix form

$$\hat{x} = \mathbf{F}^{-1} x = \frac{1}{N} \mathbf{F}^H x$$

where $\mathbf{F}^H = (\bar{\mathbf{F}})^T$ is Hermitian transpose of $\mathbf{F}$

Contents

DT Fourier Series

DT Fourier series for x with period N and $\omega_0 = \frac{2\pi}{N}$,

$$x[n] = \sum_{k \in [N]} \hat{x}[k] e^{jk\omega_0 n}$$

Correspondence between two N-periodic functions

$$x \overset{\mathcal{DTFS}}{\longleftrightarrow} \hat{x} \qquad \text{or} \qquad x[n] \overset{\mathcal{DTFS}}{\longleftrightarrow} \hat{x}[k]$$

Two equivalent representations of same signal

- time domain: $x[n]$
- frequency domain: $\hat{x}[k]$

Properties of DT Fourier Series

Linearity

If x, y have **same** period N,

$$\widehat{ax + by} = a\hat{x} + b\hat{y}$$

Time and frequency shifting

If x has period N and $\omega_0 = \frac{2\pi}{N}$,

$$\widehat{\tau_{n_0} x} = E_{-\omega_0 n_0}\hat{x} \qquad \text{or} \qquad x[n - n_0] \overset{\mathcal{DTFS}}{\longleftrightarrow} e^{-jk\omega_0 n_0}\hat{x}[k]$$

and

$$\widehat{E_{m\omega_0} x} = \tau_m \hat{x} \qquad \text{or} \qquad e^{jm\omega_0 n}x[n] \overset{\mathcal{DTFS}}{\longleftrightarrow} \hat{x}[k - m]$$

where $(E_a\hat{x})[k] = e^{jak}\hat{x}[k]$ and $(E_a x)[n] = e^{jan}x[n]$

Properties of DT Fourier Series

Assume x has period N

Time reversal

$$\widehat{Rx} = R\hat{x} \qquad \text{or} \qquad x[-n] \overset{\mathcal{DTFS}}{\longleftrightarrow} \hat{x}[-k]$$

Conjugation

$$\widehat{x^*} = R\hat{x}^* \qquad \text{or} \qquad x^*[n] \overset{\mathcal{DTFS}}{\longleftrightarrow} (\hat{x}[-k])^*$$

Symmetry

- x even $\iff$ $\hat{x}$ even, x odd $\iff$ $\hat{x}$ odd
- x real $\iff$ $\hat{x}[-k] = \overline{\hat{x}[k]}$
- x real and even $\iff$ $\hat{x}$ real and even
- x real and odd $\iff$ $\hat{x}$ purely imaginary and odd

Time Scaling

Define $x_{(m)}$ by

$$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is multiple of } m \\ 0, & \text{otherwise} \end{cases}$$

If x has period N, then $x_{(m)}$ has period mN, and

$$\widehat{x_{(m)}} = \frac{1}{m}\hat{x} \qquad \text{or} \qquad x_{(m)}[n] \overset{\mathcal{DTFS}}{\longleftrightarrow} \frac{1}{m}\hat{x}[k]$$

Proof.

$$\widehat{x_{(m)}}[k] = \frac{1}{mN}\sum_{n\in[mN]} x_{(m)}[n]e^{-jk\frac{2\pi}{mN}n} = \frac{1}{mN}\sum_{\ell\in[N]} x[\ell]e^{-jk\frac{2\pi}{N}\ell} = \frac{1}{m}\hat{x}[k]$$

NB. $x_{(m)}$ and $\widehat{x_{(m)}}$ have period mN, so $x_{(m)}[n] = \sum_{k\in[mN]} \frac{1}{m}\hat{x}[k]e^{jk\frac{2\pi}{mN}n}$

First Difference and Running Sum

First (backward) difference (analog of derivative for CT signals)

$$\Delta x = x - \tau_1 x$$

If x has period N, so does Δx, and

$$\widehat{\Delta x} = (1 - E_{-\frac{2\pi}{N}})\hat{x} \qquad \text{or} \qquad x[n] - x[n-1] \overset{\mathcal{DTFS}}{\longleftrightarrow} (1 - e^{-jk\frac{2\pi}{N}})\hat{x}[k]$$

Running sum (analog of integration for CT signals)

$$y[n] = \sum_{m=n_0}^{n} x[m]$$

- y periodic iff $\hat{x}[0] = 0$, i.e. x has no DC component
- if $\hat{x}[0] = 0$, y also has period N,

$$\hat{y}[k] = \frac{1}{1 - e^{-jk\frac{2\pi}{N}}}\hat{x}[k] \quad \text{for } k \neq 0$$

Multiplication

If x and y have **same** period N, so does their product xy, and

$$\widehat{xy} = \hat{x} * \hat{y} \qquad \text{or} \qquad x[N]y[N] \overset{\mathcal{DTFS}}{\longleftrightarrow} \sum_{m \in [N]} \hat{x}[m]\hat{y}[k-m]$$

NB. Frequency domain: **periodic** convolution in DT case vs. **aperiodic** convolution in CT case

Proof.

$$\begin{aligned} x[n]y[n] &= \left(\sum_{m \in [N]} \hat{x}[m] e^{jm\omega_0 n}\right)\left(\sum_{\ell \in [N]} \hat{y}[\ell] e^{j\ell\omega_0 n}\right) \\ &= \sum_{m \in [N]} \sum_{\ell \in [N]} \hat{x}[m]\hat{y}[\ell] e^{j(m+\ell)\omega_0 n} \\ &= \sum_{k \in [N]} \left(\sum_{m \in [N]} \hat{x}[m]\hat{y}[k-m]\right) e^{jk\omega_0 n} \qquad \left(k = m + \ell \text{, use arithmetic mod } N\right) \end{aligned}$$

Periodic Convolution

Periodic convolution $x * y$ of x and y with **same** period N

$$(x * y)[n] = \sum_{m \in [N]} x[m]y[n-m]$$

Properties

- Commutativity

$$x * y = y * x$$

- Associativity

$$(x * y) * z = x * (y * z)$$

- Bilinearity

$$\left(\sum_i a_i x_i\right) * \left(\sum_j b_j y_j\right) = \sum_{i,j} a_i b_j (x_i * y_j)$$

Periodic Convolution

Fourier coefficients satisfy

$$\widehat{x * y} = N\hat{x}\hat{y} \qquad \text{or} \qquad (x * y)[n] \overset{\mathcal{DTFS}}{\longleftrightarrow} N\hat{x}[k]\hat{y}[k]$$

convolution in time $\iff$ multiplication in frequency

Proof.

$$\begin{aligned}
(\widehat{x * y})[k] &= \langle x * y, e^{jk\omega_0 n} \rangle \\
&= \langle \sum_{m \in [N]} x[m] \tau_m y, e^{jk\omega_0 n} \rangle \\
&= \sum_{m \in [N]} x[m] \langle \tau_m y, e^{jk\omega_0 n} \rangle \\
&= \sum_{m \in [N]} x[m] \hat{y}[k] e^{-jk\omega_0 m} = N\hat{x}[k]\hat{y}[k]
\end{aligned}$$