EI331 Signals and Systems

Lecture 14

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Contents

CT Fourier Transform

Fourier transform (analysis equation)

$$X(j\omega) = \mathcal{F}\{x\}(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$X(j\omega)$ called spectrum of $x(t)$

Inverse Fourier transform (synthesis equation)

$$x(t) = \mathcal{F}^{-1}\{X\}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Superposition of complex exponentials at **continuum** of frequencies; frequency ω has density $\frac{1}{2\pi}X(j\omega)$

CT Fourier Transform

Two equivalent representations of same signal

- time domain vs. frequency domain: $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$

Also widely used elsewhere. In probability theory,

- characteristic function of random variable X with density $p(x)$

$$\varphi_X(t) = \mathbb{E}[e^{jtX}] = \int_{-\infty}^{\infty} p(x) e^{jxt} dx$$

In quantum mechanics

- position representation: wave function $\psi(x)$
  - $|\psi(x)|^2$ probability density of finding particle at position x
- momentum representation: $\Psi(p)$

$$\Psi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) e^{-jpx/\hbar} dx$$

  - $|\Psi(p)|^2$ probability density of finding particle with momentum p

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Fourier Transform of L_1 Signals

Recall from calculus, for real-valued x, improper integral

$$\int_{\mathbb{R}} x(t)dt \triangleq \lim_{T_1,T_2\to\infty} \int_{-T_1}^{T_2} x(t)dt$$

is well-defined if $x \in L_1(\mathbb{R})$, i.e. $\|x\|_1 = \int_{\mathbb{R}} |x(t)|dt < \infty$

Same is true for complex-valued $x = u + jv$, since $|u|, |v| \leq |x|$ and

$$\int_{\mathbb{R}} x(t)dt \triangleq \int_{\mathbb{R}} u(t)dt + j\int_{\mathbb{R}} v(t)dt$$

For signal $x \in L_1(\mathbb{R})$

- $x(t)e^{-j\omega t} \in L_1(\mathbb{R})$, since $|x(t)e^{-j\omega t}| = |x(t)|$
- Fourier transform $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ is well-defined improper integral for each ω

Fourier Transform of L_1 Signals

Theorem. If $x \in L_1$, then X is bounded, in fact $\|X\|_\infty \leq \|x\|_1$

Proof. Consequence of following lemma.

Lemma. For complex-valued function f of real variable t,

$$\left| \int f(t) dt \right| \leq \int |f(t)| dt$$

Proof. If $\int |f(t)| dt = \infty$, trivial. Assume $\int |f(t)| dt < \infty$. Let phase of $\int f(t) dt$ be ϕ, i.e. $\int f(t) dt = | \int f(t) dt | e^{j\phi}$. Then

$$\left| \int f(t) dt \right| = e^{-j\phi} \int f(t) dt = \int e^{-j\phi} f(t) dt$$

Taking real part

$$\left| \int f(t) dt \right| = \text{Re} \int e^{-j\phi} f(t) dt = \int \text{Re} [e^{-j\phi} f(t)] dt \leq \int |e^{-j\phi} f(t)| dt$$

Fourier Transform of L_1 Signals

Theorem. If $x \in L_1$, then X is uniformly continuous

Proof. For any $T > 0$,

$$
\begin{aligned}
|X(j\omega_1) - X(j\omega_2)| &\le \int_{\mathbb{R}} |x(t)e^{-j\omega_1 t} - x(t)e^{-j\omega_2 t}| dt \\
&= \int_{\mathbb{R}} |x(t)| \cdot \left| 2 \sin \frac{\Delta\omega t}{2} \right| dt \quad (\Delta\omega = \omega_1 - \omega_2) \\
&\le 2 \int_{|t|>T} |x(t)| dt + |\Delta\omega| T \int_{|t|\le T} |x(t)| dt \\
&\le 2 \int_{|t|>T} |x(t)| dt + |\Delta\omega| T \|x\|_1 \triangleq I_1 + I_2
\end{aligned}
$$

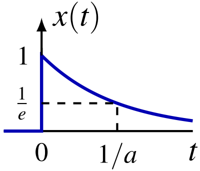
Given $\epsilon > 0$, since $x \in L_1(\mathbb{R})$, exists $T > 0$ s.t. $I_1 < \epsilon/2$.

Fix such T. For $|\Delta\omega| \le \epsilon/(2T\|x\|_1)$, $I_2 < \epsilon/2$.

Thus $|X(j\omega_1) - X(j\omega_2)| < \epsilon$.

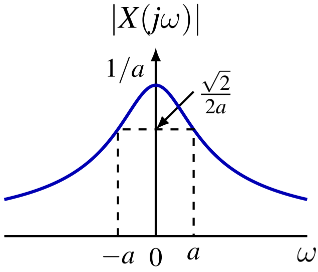
Example: Right-sided Decaying Exponential

$$x(t) = e^{-at}u(t), \quad a > 0$$



$$X(j\omega) = \int_0^\infty e^{-(a+j\omega)t}dt$$

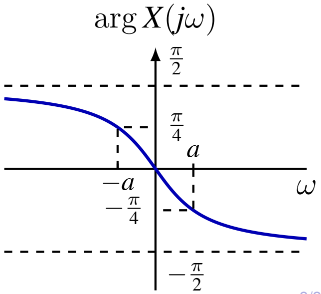
$$= -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^\infty = \frac{1}{a+j\omega}$$



NB. Above formula for X also works for complex a with Re $a > 0$.

For $a > 0$,
$$|X(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}},$$

$$\arg X(j\omega) = -\arctan\frac{\omega}{a}$$

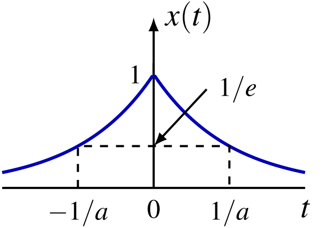


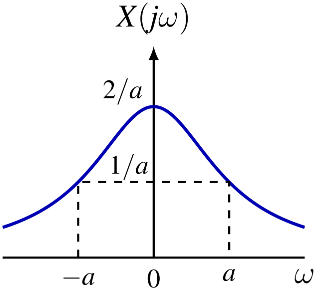
Example: Two-sided Decaying Exponential

$$x(t) = e^{-a|t|}, \quad a > 0$$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt \\ &= \int_{-\infty}^{0} e^{(a-j\omega)t} dt + \int_{0}^{\infty} e^{-(a+j\omega)t} dt \\ &= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} \\ &= \frac{2a}{a^2+\omega^2} \end{aligned}$$

NB. Above formula for X also works for complex a with Re $a > 0$.

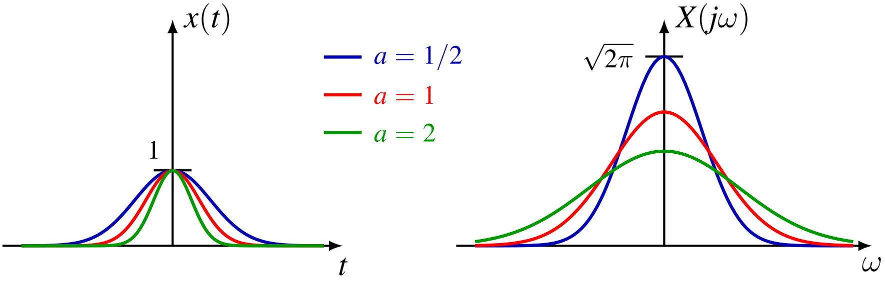




Example: Gaussian

For $a > 0$,

$$x(t) = e^{-at^2} \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}}$$



In particular, $x(t) = e^{-\frac{1}{2}t^2} \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \sqrt{2\pi} e^{-\frac{1}{2}\omega^2} = \sqrt{2\pi} x(\omega)$,
i.e. $\mathcal{F}\{x\} = \sqrt{2\pi} x$

Example: Gaussian

For $a > 0$,

$$x(t) = e^{-at^2} \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}}$$

Proof.

$$\begin{aligned}
\frac{d}{d\omega} X(j\omega) &= \int_{-\infty}^{\infty} e^{-at^2} (-jt) e^{-j\omega t} dt = \frac{j}{2a} \int_{-\infty}^{\infty} \left(\frac{d}{dt} e^{-at^2} \right) e^{-j\omega t} dt \\
&= -\frac{j}{2a} \int_{-\infty}^{\infty} e^{-at^2} \left(\frac{d}{dt} e^{-j\omega t} \right) dt \quad \text{(integration by parts)} \\
&= -\frac{\omega}{2a} \int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t} dt = -\frac{\omega}{2a} X(j\omega)
\end{aligned}$$

$$\frac{d}{d\omega} \left(X(j\omega) e^{\frac{\omega^2}{4a}} \right) = 0 \implies X(j\omega) = X(j0) e^{-\frac{\omega^2}{4a}} = \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}}$$

Fourier Inversion for L_1 Signals

Given Fourier transform,

$$X(j\omega) = \mathcal{F}\{x\}(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Is inverse Fourier transform well-defined? Is it equal to x?

$$x(t) \overset{?}{=} \mathcal{F}^{-1}\{X\}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Theorem. If $x \in L_1(\mathbb{R})$ is continuous and $X = \mathcal{F}\{x\} \in L_1(\mathbb{R})$, then $x = \mathcal{F}^{-1}\{X\}$.

- e.g. Two-sided decaying exponential, Gaussian

But, for $x(t) \in L_1(\mathbb{R})$, $X(j\omega)$ is **not** necessarily in $L_1(\mathbb{R})$

- e.g. one-sided decaying exponential, rectangular pulse
- if $X \in L_1(\mathbb{R})$, x must be continuous

Fourier Inversion for L_1 Signals

Inverse FT typically interpreted as principal value, i.e.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \lim_{W\to\infty}\frac{1}{2\pi}\int_{-W}^{W} X(j\omega)e^{j\omega t}d\omega$$

may converge without being absolutely convergent

Theorem. If $x \in L_1(\mathbb{R})$ satisfies Dirichlet conditions on all finite intervals, then

$$\lim_{W\to\infty}\frac{1}{2\pi}\int_{-W}^{W} X(j\omega)e^{j\omega t}d\omega = \frac{x(t_+)+x(t_-)}{2} \quad \text{pointwise}$$

NB. Gibbs phenomenon at discontinuity

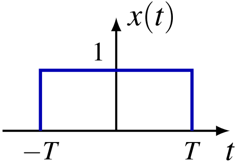
Often also need to interpret FT as principal value

$$\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \lim_{T\to\infty}\int_{-T}^{T} x(t)e^{-j\omega t}dt$$

Example: Rectangular Pulse

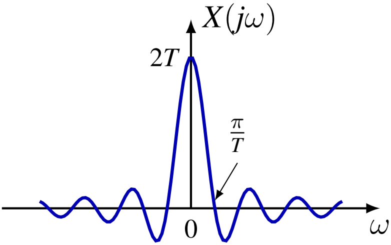
$$x(t) = u(t+T) - u(t-T)$$

$$X(j\omega) = \int_{-T}^{T} e^{-j\omega t} dt = \frac{2\sin(\omega T)}{\omega}$$



Inverse FT

$$\int_{-\infty}^{\infty} \frac{\sin(\omega T)}{\pi\omega} d\omega = x(0) = 1$$



As $T \to \infty$,

- frequency domain

$$\lim_{T\to\infty} \frac{\sin(\omega T)}{\pi\omega} = \delta(\omega)$$

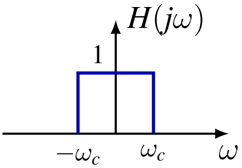
- time domain: $x(t) \to 1$, DC ✓

$$\text{sinc}(\theta) \triangleq \frac{\sin(\pi\theta)}{\pi\theta}$$

Example: Ideal Lowpass Filter

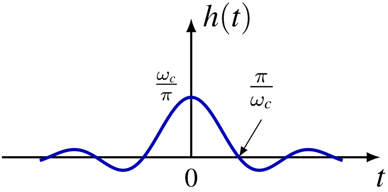
Frequency response

$$H(j\omega) = u(\omega + \omega_c) - u(\omega - \omega_c)$$



Impulse response

$$h(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t} d\omega = \frac{\sin(\omega_c t)}{\pi t}$$

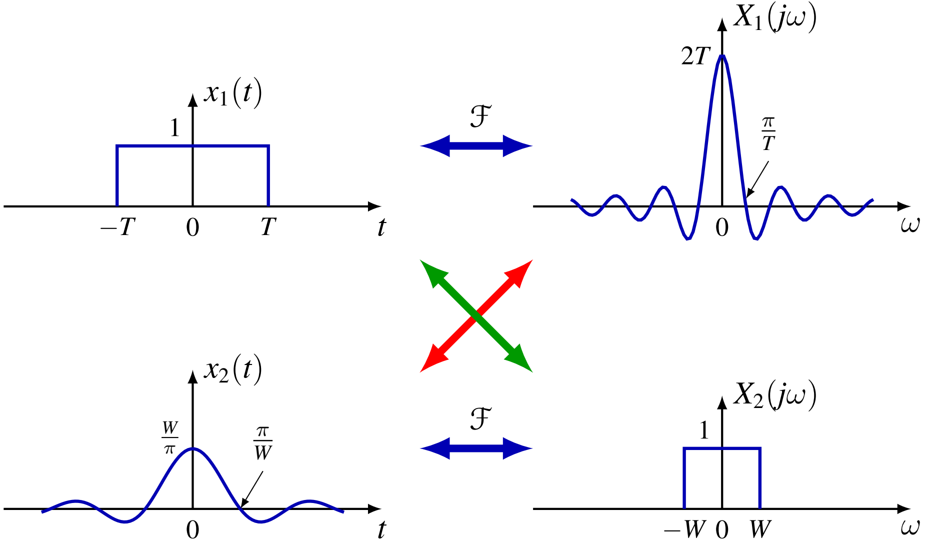


NB. $h(t) \notin L_1(\mathbb{R})$ but $H(j\omega) \in L_1(\mathbb{R})$

As $\omega_c \to \infty$,

- time domain
  - $h(t) \to \delta(t)$, becomes identity system
- frequency domain
  - $H(j\omega) \to 1$, passes all frequencies ✓

Duality



$x_1(t)$
1
$-T$
0
T
t
$\mathcal{F}$
$X_1(j\omega)$
$2T$
$\frac{\pi}{T}$
0
ω
$x_2(t)$
$\frac{W}{\pi}$
$\frac{\pi}{W}$
0
t
$\mathcal{F}$
$X_2(j\omega)$
1
$-W$
0
W
ω

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Fourier Transform of More General Functions

If $x_n \to x$, define Fourier transform of x by

$$X(j\omega) = \mathcal{F}\{x\} \triangleq \lim_n \mathcal{F}\{x_n\}$$

i.e.

$$X(j\omega) = \int_{\mathbb{R}} x(t)e^{-j\omega t}dt \triangleq \lim_n \int_{\mathbb{R}} x_n(t)e^{-j\omega t}dt$$

If $x \in L_1(\mathbb{R})$, above definition is consistent with old one

In general, convergence interpreted in **distributional** sense, i.e. for nice test function ϕ

$$\int_{\mathbb{R}} X(j\omega)\phi(\omega)d\omega \triangleq \lim_n \int_{\mathbb{R}} \left(\int_{\mathbb{R}} x_n(t)e^{-j\omega t}dt \right) \phi(\omega)d\omega$$

Interchanging order of integration leads to alternative definition

$$\int_{\mathbb{R}} X(j\omega)\phi(\omega)d\omega = \lim_n \int_{\mathbb{R}} x_n(t) \left(\int_{\mathbb{R}} \phi(\omega)e^{-j\omega t}d\omega \right) dt = \int_{\mathbb{R}} x(t)\Phi(jt)dt$$

Schwarz Space

Space of test functions is so-called Schwarz space on $\mathbb{R}$, denoted $\mathcal{S} = \mathcal{S}(\mathbb{R})$

Function $\phi \in \mathcal{S}$ if it is

- infinitely differentiable: $\phi^{(k)}$ exists for all $k \in \mathbb{N}$
- rapidly decreasing:

$$\|\phi\|_{\ell,k} \triangleq \sup_{t\in\mathbb{R}} |t^\ell \phi^{(k)}(t)| < \infty, \quad \forall \ell, k \in \mathbb{N}$$

Example. Gaussian $g(t) = e^{-at^2} \in \mathcal{S}$

Note $\phi \in \mathcal{S} \implies \phi \in L_1$, Fourier transform Φ well-defined.

Theorem. If $\phi \in \mathcal{S}$, then $\Phi \in \mathcal{S}$

Example. Gaussian $G(j\omega) = \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}} \in \mathcal{S}$

Example: Unit Impulse δ

Method 1. Recall for ideal lowpass filter,

$$h_W(t) = \frac{\sin(Wt)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} H_W(j\omega) = u(\omega + W) - u(\omega - W)$$

Since $h_W \to \delta$ as $W \to \infty$

$$\mathcal{F}\{\delta\} = \lim_{W\to\infty} H_W(j\omega) = 1$$

Can also use Gaussian instead of sinc. Recall

$$g_a(t) = \frac{1}{\sqrt{\pi a}} e^{-\frac{t^2}{a}} \overset{\mathcal{F}}{\longleftrightarrow} G_a(j\omega) = e^{-\frac{a\omega^2}{4}}$$

Since $g_a \to \delta$ as $a \to 0$

$$\mathcal{F}\{\delta\} = \lim_{a\to 0} G_a(j\omega) = 1$$

Example: Unit Impulse δ

Method 1. In fact, for any $x_n \to \delta$,

$$\mathcal{F}(\delta) = \lim_n \int x_n(t) e^{-j\omega t} dt = \int_{\mathbb{R}} \delta(t) e^{-j\omega t} dt = e^{-j\omega \cdot 0} = 1$$

Method 2. Let $X = \mathcal{F}\{\delta\}$.

$$\int_{\mathbb{R}} X(j\omega)\phi(\omega) d\omega = \int_{\mathbb{R}} \delta(t)\Phi(jt) dt = \Phi(0) = \int_{\mathbb{R}} 1 \cdot \phi(\omega) d\omega$$

Thus $X(j\omega) = 1$.

δ has "white" spectrum, equal amount of all frequencies!

Inverse Fourier transform

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\omega = \lim_{W \to \infty} \frac{1}{2\pi} \int_{-W}^{W} e^{j\omega t} d\omega = \lim_{W \to \infty} \frac{\sin(Wt)}{\pi t}$$

Example: DC Signal 1

Method 1. Recall for rectangular pulse,

$$x_T(t) = u(t+T) - u(t-T) \overset{\mathcal{F}}{\longleftrightarrow} X_T(j\omega) = \frac{2\sin(\omega T)}{\omega}$$

Since $x_T \to 1$ as $T \to \infty$

$$\mathcal{F}\{1\} = \lim_{T\to\infty} X_T(j\omega) = 2\pi \lim_{T\to\infty} \frac{\sin(\omega T)}{\pi\omega} = 2\pi\delta(\omega)$$

NB. Same as direct calculation using FT formula

Can also use Gaussian instead of rectangular pulse. Recall

$$\tilde{g}_a(t) = e^{-at^2} \overset{\mathcal{F}}{\longleftrightarrow} \tilde{G}_a(j\omega) = \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}}$$

Since $\tilde{g}_a \to 1$ as $a \to 0$ and $\{\tilde{G}_a\}$ is family of good kernels

$$\mathcal{F}\{1\} = \lim_{a\to 0} \tilde{G}_a(j\omega) = \delta(\omega)$$

Example: DC Signal 1

Method 2. Let $X = \mathcal{F}\{1\}$.

$$\int_{\mathbb{R}} X(j\omega)\phi(\omega)d\omega = \int_{\mathbb{R}} 1 \cdot \Phi(jt)dt = 2\pi\phi(0) = \int_{\mathbb{R}} 2\pi\delta(\omega)\phi(\omega)d\omega$$

Thus $X(j\omega) = 2\pi\delta(\omega)$. Formally

$$2\pi\delta(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}dt$$

Spectrum of DC signal is impulse at zero frequency!

Inverse Fourier transform

$$1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega)e^{j\omega t}d\omega$$

Example: Complex Exponentials

Let $X = \mathcal{F}\{x\}$ for $x(t) = e^{j\omega_0 t}$.

$$\int_{\mathbb{R}} X(j\omega)\phi(\omega)d\omega = \int_{\mathbb{R}} e^{j\omega_0 t}\Phi(jt)dt = 2\pi\phi(\omega_0)$$
$$= \int_{\mathbb{R}} 2\pi\delta(\omega - \omega_0)\phi(\omega)d\omega$$

Thus $X(j\omega) = 2\pi\delta(\omega - \omega_0)$. Formally,

$$2\pi\delta(\omega - \omega_0) = \int_{\mathbb{R}} e^{j(\omega_0 - \omega)t}dt$$

Spectrum of $e^{j\omega_0 t}$ is impulse at ω_0!

Inverse Fourier transform

$$e^{j\omega_0 t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega$$

Example: Complex Exponentials

$$2\pi\delta(\omega_1 - \omega_2) = \int_{\mathbb{R}} e^{j(\omega_1 - \omega_2)t} dt = \int_{\mathbb{R}} e^{j\omega_1 t} \overline{e^{j\omega_2 t}} dt$$

Can be interpreted as orthogonality of $e^{j\omega t}$

- $e^{j\omega_1 t}$ and $e^{j\omega_2 t}$ are orthogonal if $\omega_1 \neq \omega_2$

CT Fourier transform can be considered as “orthogonal” expansion into continuum of “basis” functions

$$X(j\omega) = \langle x, e^{j\omega t} \rangle = \int_{\mathbb{R}} x(t) \overline{e^{j\omega t}} dt$$

and

$$x(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \langle x, e^{j\omega t} \rangle e^{j\omega t} d\omega$$

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Fourier Transform of Periodic Signals

Periodic signal with fundamental frequency ω_0 has Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} \hat{x}[k] e^{jk\omega_0 t}$$

Fourier transform

$$X(j\omega) = \int_{\mathbb{R}} x(t) e^{-j\omega t} dt = \int_{\mathbb{R}} \sum_{k=-\infty}^{\infty} \hat{x}[k] e^{jk\omega_0 t} e^{-j\omega t} dt$$

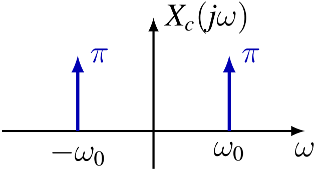
$$= \sum_{k=-\infty}^{\infty} \hat{x}[k] \int_{\mathbb{R}} e^{jk\omega_0 t} e^{-j\omega t} dt = \sum_{k=-\infty}^{\infty} 2\pi \hat{x}[k] \delta(\omega - k\omega_0)$$

Spectrum of periodic signal consists of impulses at harmonically related frequencies! Areas of impulses are 2π times Fourier series coefficients

Example: Sine and Cosine

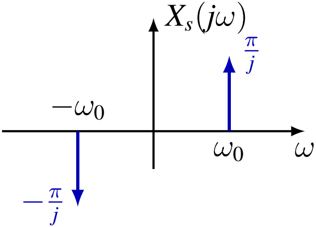
$$x_c(t) = \cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

$$X_c(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

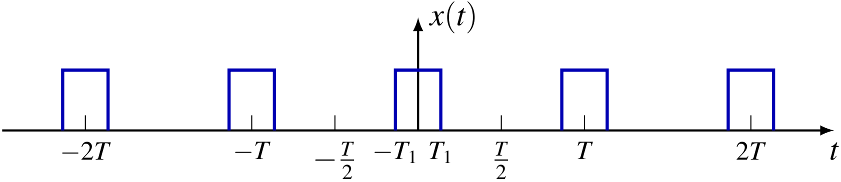


$$x_s(t) = \sin(\omega_0 t) = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$$

$$X_s(j\omega) = \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$

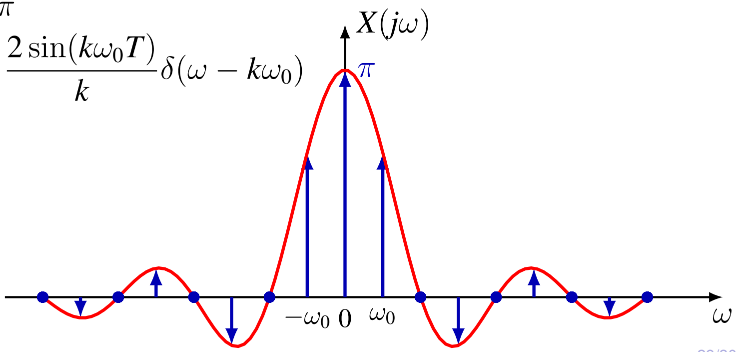


Example: Periodic Square Wave



$$\hat{x}[k] = \frac{\sin(k\omega_0 T)}{k\pi}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\sin(k\omega_0 T)}{k} \delta(\omega - k\omega_0)$$



Example: Periodic Impulse Train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta\left(\omega - \frac{2k\pi}{T}\right)$$