EI331 Signals and Systems

Lecture 16

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Contents

Parseval's Identity

Theorem. If $x \in L_2(\mathbb{R})$, $X = \mathcal{F}\{x\}$, then

$$\|x\|_2^2 = \frac{1}{2\pi}\|X\|_2^2, \quad \text{or} \quad \int_{\mathbb{R}} |x(t)|^2 dt = \frac{1}{2\pi}\int_{\mathbb{R}} |X(j\omega)|^2 d\omega$$

Note ω is angular frequency and $\frac{\omega}{2\pi}$ is frequency

$$\int_{\mathbb{R}} |x(t)|^2 dt = \int_{\mathbb{R}} |X(j\omega)|^2 \frac{d\omega}{2\pi}$$

Interpretation: Energy conservation

- $|x(t)|^2$ power, or energy per unit time (second)
- $|X(j\omega)|^2$ energy per unit frequency (Hertz)

$|X(j\omega)|^2$ called energy-density spectrum

Parseval's Identity

Theorem. If $x, y \in L_2(\mathbb{R})$, $X = \mathcal{F}\{x\}$, $Y = \mathcal{F}\{y\}$, then

$$\langle x, y \rangle = \frac{1}{2\pi}\langle X, Y \rangle, \qquad \text{or} \qquad \int_{\mathbb{R}} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{\mathbb{R}} X(j\omega)Y^*(j\omega)d\omega$$

"Proof."

$$\begin{aligned}
\int_{\mathbb{R}} x(t)y^*(t)dt &= \int_{\mathbb{R}} x(t)\left(\frac{1}{2\pi}\int_R Y(j\omega)e^{j\omega t}d\omega\right)^* dt \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} x(t)\int_R Y^*(j\omega)e^{-j\omega t}d\omega dt \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} x(t)e^{-j\omega t}dt\right)Y^*(j\omega)d\omega \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} X(j\omega)Y^*(j\omega)d\omega
\end{aligned}$$

Parseval's Identity

Example.

$$\frac{\sin(Wt)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} u(\omega + W) - u(\omega - W)$$

By Parseval's identity

$$\begin{aligned}
\int_{\mathbb{R}} \frac{\sin^2(Wt)}{\pi^2 t^2} dt &= \frac{1}{2\pi} \int_{\mathbb{R}} |u(\omega + W) - u(\omega - W)|^2 d\omega \\
&= \frac{1}{2\pi} \int_{-W}^{W} d\omega \\
&= \frac{W}{\pi}
\end{aligned}$$

Contents

Convolution Property

For LTI systems T with

- impulse response h
- frequency response $H(j\omega) = \int_{\mathbb{R}} h(t)e^{-j\omega t}dt = \mathcal{F}\{h\}$

$e^{j\omega t}$ is eigenfunction associated with eigenvalue $H(j\omega)$

$$T(e^{j\omega t}) = H(j\omega)e^{\omega t}$$

Input x is linear superposition of $e^{j\omega t}$

$$x(t) = \frac{1}{2\pi}\int_{\mathbb{R}} X(j\omega)e^{j\omega t}d\omega$$

Output

$$\begin{aligned} y(t) = (x * h)(t) &= T\left(\frac{1}{2\pi}\int_{\mathbb{R}} X(j\omega)e^{j\omega t}d\omega\right) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} X(j\omega)T(e^{j\omega t})d\omega = \frac{1}{2\pi}\int_{\mathbb{R}} X(j\omega)H(j\omega)e^{j\omega t}d\omega \end{aligned}$$

Convolution Property

$$\mathcal{F}\{x * y\} = \mathcal{F}\{x\}\mathcal{F}\{y\}, \quad \text{or} \quad (x * y)(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)Y(j\omega)$$

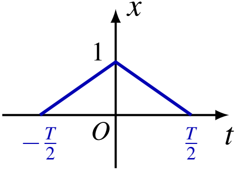
convolution in time $\iff$ multiplication in frequency

"Proof".

$$\begin{aligned}
\int_{\mathbb{R}} (x * y)(t) e^{-j\omega t} dt &= \int_{\mathbb{R}} \int_{\mathbb{R}} x(\tau) y(t-\tau) e^{-j\omega t} d\tau dt \\
&= \int_{\mathbb{R}} x(\tau) \left(\int_{\mathbb{R}} y(t-\tau) e^{-j\omega t} dt \right) d\tau \\
&= \int_{\mathbb{R}} x(\tau) Y(j\omega) e^{-j\omega\tau} d\tau \\
&= Y(j\omega) \int_{\mathbb{R}} x(\tau) e^{-j\omega\tau} d\tau \\
&= Y(j\omega) X(j\omega)
\end{aligned}$$

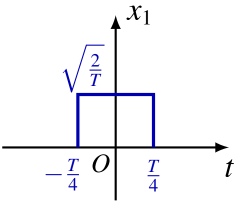
Example

$$x(t) = \left(1 - \frac{2|t|}{T}\right)\left[u(t + \frac{T}{2}) - u(t - \frac{T}{2})\right]$$

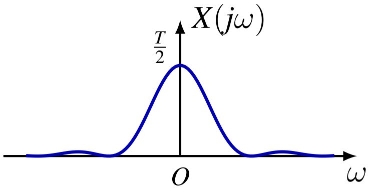


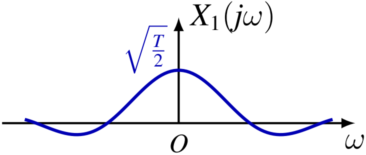
$$x = x_1 * x_1$$

$$X_1(j\omega) = \sqrt{\frac{2}{T}} \cdot \frac{2\sin\left(\frac{\omega T}{4}\right)}{\omega}$$



$$X(j\omega) = X_1^2(j\omega) = \frac{8\sin^2\left(\frac{\omega T}{4}\right)}{\omega^2 T}$$





Frequency Response of LTI System

LTI system T

- fully characterized by impulse response h

$$y = T(x) = h * x$$

- also fully characterized by frequency response $H = \mathcal{F}\{h\}$, if H is well-defined
  - BIBO stable system, $h \in L_1(\mathbb{R})$
  - other systems: identity $h = \delta$, differentiator $h = \delta'$, ...

Typically, convolution property implies

$$Y = \mathcal{F}\{y\} = HX = \mathcal{F}\{h\}\mathcal{F}\{x\}$$

Instead of computing $x * h$, can do

$$y = \mathcal{F}^{-1}(\mathcal{F}\{h\}\mathcal{F}\{x\})$$

Frequency Response of LTI System

	$h(t)$	$H(j\omega)$
id	$\delta(t)$	1
τ_{t_0}	$\delta(t-t_0)$	$e^{-j\omega t_0}$
$\frac{d}{dt}$	$\delta'(t)$	$j\omega$
$\int_{-\infty}^{t}$	$u(t)$	$\frac{1}{j\omega}+\pi\delta(\omega)$
ideal lowpass	$\frac{\sin(\omega_c t)}{\pi t}$	$u(\omega+\omega_c)-u(\omega-\omega_c)$
1st order lowpass	$\frac{1}{\tau}e^{-t/\tau}u(t)$	$\frac{1}{1+j\tau\omega}$

Examples

Example. Differentiation property

$$y = x' = x * \delta'$$

$$Y(j\omega) = X(j\omega)\mathcal{F}\{\delta'\} = j\omega X(j\omega)$$

Example. Integration property

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau = (x * u)(t)$$

$$Y(j\omega) = X(j\omega)U(j\omega) = X(j\omega)\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] = \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

Example

Unit ramp function $u_{-2}(t) = tu(t) = (u * u)(t)$

Convolution property suggests

$$\mathcal{F}\{u_{-2}\}(j\omega) = U^2(j\omega) = -\frac{1}{\omega^2} + \frac{2\pi}{j\omega}\delta(\omega) + \pi^2\delta(\omega)\delta(\omega)$$

But $\frac{\pi}{j\omega}\delta(\omega)$ and $\delta(\omega)\delta(\omega)$ **not well-defined!**

Know $\mathcal{F}\{u_{-2}\} = -\frac{1}{\omega^2} + j\pi\delta'(\omega)$

Convolution property **not applicable** here!

Rule of thumb. Applicable when formula is well-defined

Example

Response of LTI system with impulse response $h(t) = e^{-at}u(t)$ to input $x(t) = e^{-bt}u(t)$, $a, b > 0$

Method 1. Direct convolution $y = x * h$

Method 2. Solve following ODE with initial rest condition

$$y'(t) + ay(t) = e^{-bt}u(t)$$

Method 3. Fourier transform.

$$H(j\omega) = \frac{1}{a + j\omega},\ X(j\omega) = \frac{1}{b + j\omega} \implies Y(j\omega) = \frac{1}{(a + j\omega)(b + j\omega)}$$

If $a \neq b$, $Y(j\omega) = \frac{1}{b-a}(\frac{1}{a+j\omega} - \frac{1}{b+j\omega}) \implies y(t) = \frac{1}{b-a}(e^{-at} - e^{-bt})u(t)$

If $a = b$, $Y(j\omega) = -\frac{d}{da}\left(\frac{1}{a+j\omega}\right) \implies y(t) = -\frac{d}{da}e^{-at}u(t) = te^{-at}u(t)$

Can also use $Y(j\omega) = j\frac{d}{d\omega}\left(\frac{1}{a+j\omega}\right)$ and differentiation property

Example

Response of LTI system with impulse response $h(t) = e^{-at}u(t)$ to input $x(t) = \cos(\omega_0 t)$, $a > 0$.

Frequency response

$$H(j\omega) = \frac{1}{a + j\omega} = \frac{1}{\sqrt{a^2 + \omega^2}} e^{-j \arctan \frac{\omega}{a}}$$

Method 1. Use eigenfunction property

$$x(t) = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$$

$$\begin{aligned} y(t) &= \frac{1}{2} H(j\omega_0) e^{j\omega_0 t} + \frac{1}{2} H(-j\omega_0) e^{-j\omega_0 t} = \text{Re} \frac{e^{j\omega_0 t}}{a + j\omega_0} \\ &= \frac{a \cos(\omega_0 t) + \omega_0 \sin(\omega_0 t)}{a^2 + \omega_0^2} = \frac{1}{\sqrt{a^2 + \omega_0^2}} \cos(\omega_0 t - \arctan \frac{\omega_0}{a}) \end{aligned}$$

Example

Response of LTI system with impulse response $h(t) = e^{-at}u(t)$ to input $x(t) = \cos(\omega_0 t)$.

Frequency response

$$H(j\omega) = \frac{1}{a + j\omega} = \frac{1}{\sqrt{a^2 + \omega^2}} e^{-j \arctan \frac{\omega}{a}}$$

Method 2. Use Fourier transform of X

$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$Y(j\omega) = \pi H(j\omega_0)\delta(\omega - \omega_0) + \pi H(-j\omega_0)\delta(\omega + \omega_0)$$

$$y(t) = \frac{1}{2} H(j\omega_0) e^{j\omega_0 t} + \frac{1}{2} H(-j\omega_0) e^{-j\omega_0 t}$$

Example

Response of ideal lowpass filter with impulse response $h(t)$ to input $x(t)$.

$$h(t) = \frac{\sin(\omega_c t)}{\pi t}, \quad x(t) = \frac{\sin(\omega_i t)}{\pi t}$$

Fourier transforms

$$X(j\omega) = u(\omega + \omega_i) - u(\omega - \omega_i)$$
$$H(j\omega) = u(\omega + \omega_c) - u(\omega - \omega_c)$$

Use $Y(j\omega) = X(j\omega)H(j\omega)$,

$$Y(j\omega) = \begin{cases} X(j\omega), & \text{if } \omega_i \leq \omega_c \\ H(j\omega), & \text{if } \omega_i > \omega_c \end{cases} \implies y(t) = \begin{cases} x(t), & \text{if } \omega_i \leq \omega_c \\ h(t), & \text{if } \omega_i > \omega_c \end{cases}$$

NB. Convolution of two sinc is another sinc

Example

Convolution of Gaussians is another Gaussian

$$x_i(t) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{(t-\mu_i)^2}{2\sigma_i^2}\right)$$

Fourier transform (complex conjugate of characteristic function)

$$X_i(j\omega) = \exp\left(-i\mu_i\omega - \frac{\sigma_i^2}{2}\omega^2\right)$$

For $y = x_1 * x_2$,

$$Y(j\omega) = X_1(j\omega)X_2(j\omega) = \exp\left(-i(\mu_1+\mu_2)\omega - \frac{\sigma_1^2+\sigma_2^2}{2}\omega^2\right)$$

$$y(t) = \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} \exp\left(-\frac{(t-\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}\right)$$

System Connections

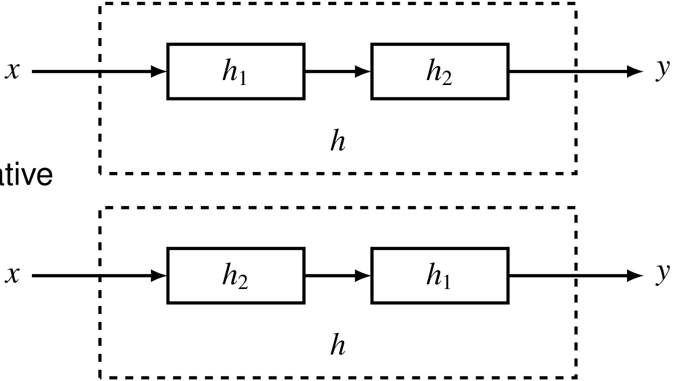
LTI systems in series connection

$$y = (x * h_1) * h_2 = x * (h_1 * h_2) = (x * h_2) * h_1$$

$h = h_1 * h_2$

commutative

$h = h_2 * h_1$

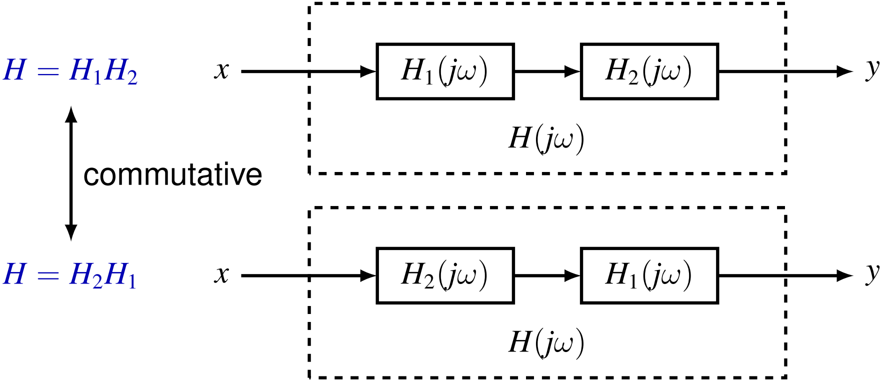


Order of processing **usually** not important for LTI systems

System Connections

LTI systems in series connection

$$Y = (XH_1)H_2 = X(H_1H_2) = (XH_2)H_1$$

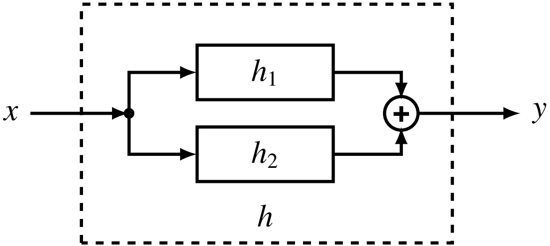


Order of processing **usually** not important for LTI systems

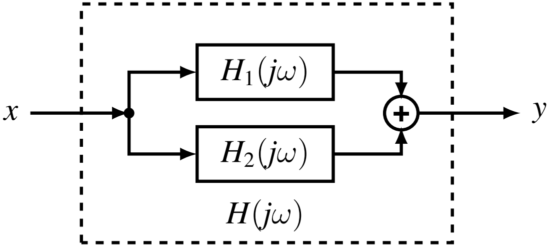
System Connections

LTI systems in systems in parallel connection

$h = h_1 + h_2$



$H = H_1 + H_2$

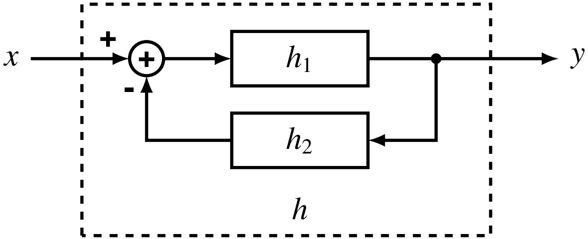


System Connections

LTI systems in systems in feedback connection

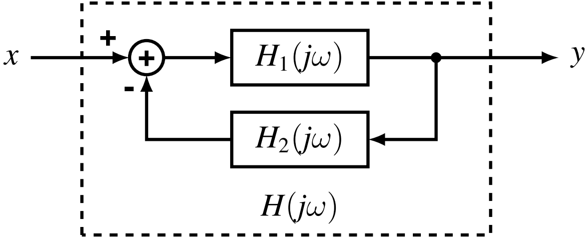
$$y = h_1 * (x - h_2 * y)$$

$$h = ?$$



$$Y = H_1 X - H_1 H_2 Y$$

$$H = \frac{Y}{X} = \frac{H_1}{1 + H_1 H_2}$$



Contents

Multiplication Property

Dual of convolution property

$$\mathcal{F}\{xy\} = \frac{1}{2\pi}\mathcal{F}\{x\}*\mathcal{F}\{y\}, \text{ or } x(t)y(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\int_{\mathbb{R}} X(j\theta)Y(j(\omega-\theta))d\theta$$

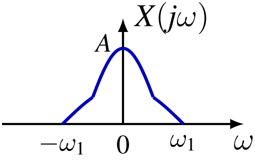
multiplication in time $\iff$ convolution in frequency

Proof. Let $Z(j\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} X(j\theta)Y(j(\omega-\theta))d\theta$

$$\begin{aligned}
\frac{1}{2\pi}\int_{\mathbb{R}} Z(j\omega)e^{j\omega t}d\omega &= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{2\pi}\left(\int_{\mathbb{R}} X(j\theta)Y(j(\omega-\theta))d\theta\right)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} X(j\theta)\frac{1}{2\pi}\left(\int_{\mathbb{R}} Y(j(\omega-\theta))e^{j\omega t}d\omega\right)d\theta \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} X(j\theta)y(t)e^{j\theta t}d\theta = x(t)y(t)
\end{aligned}$$

Example: Modulation

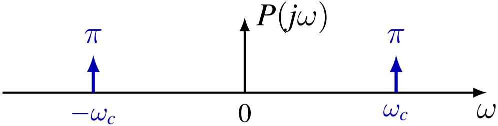
Baseband signal $x(t)$



Carrier

$$p(t) = \cos(\omega_c t)$$

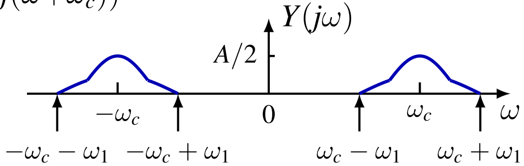
$$P(j\omega) = \pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c)$$



Modulated signal

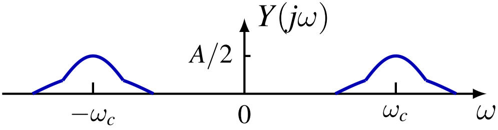
$$y(t) = x(t)p(t)$$

$$Y(j\omega) = \frac{1}{2}X(j(\omega - \omega_c)) + \frac{1}{2}X(j(\omega + \omega_c))$$



Example: Demodulation

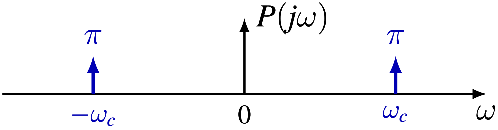
Modulated signal $y(t)$



Carrier

$p(t) = \cos(\omega_c t)$

$P(j\omega) = \pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c)$



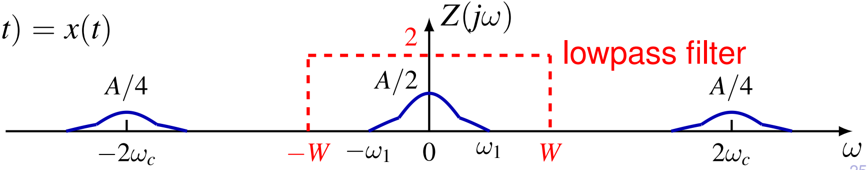
Demodulation

$$z(t) = y(t)p(t)$$

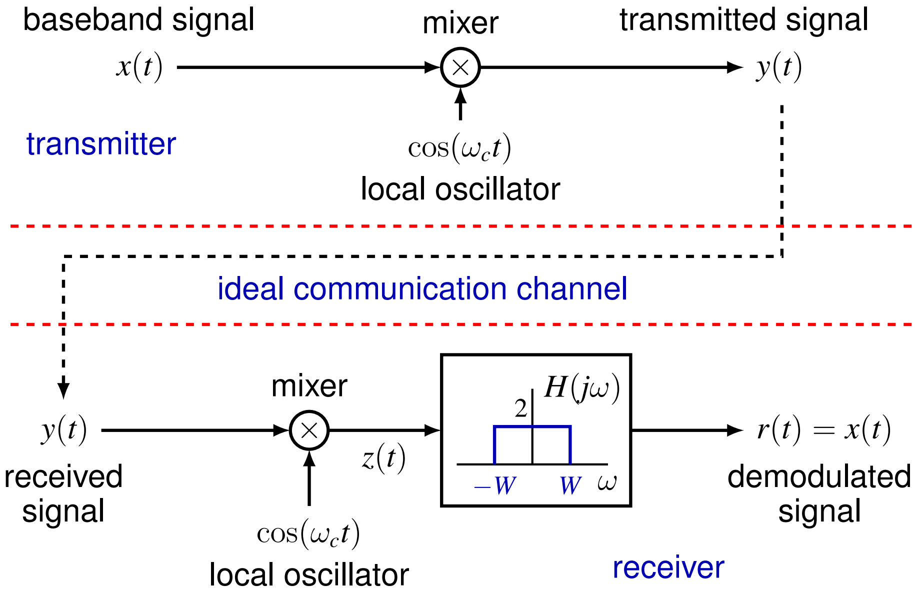
$$= \frac{1}{2}x(t) + \frac{1}{2}x(t)\cos(2\omega_c t)$$

$$R(j\omega) = Z(j\omega)H_{\text{lowpass}}(j\omega)$$

$$r(t) = x(t)$$



Ideal AM Communication System



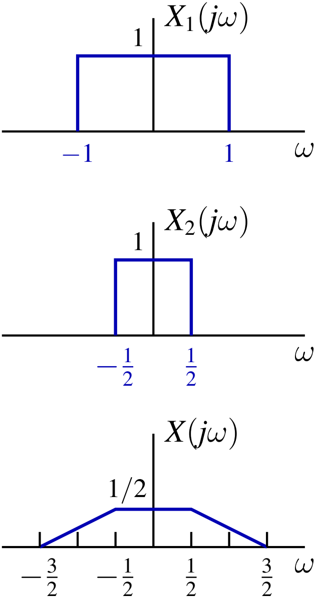
Example

$$x(t) = \frac{\sin t \cdot \sin \frac{t}{2}}{\pi t^2} = \pi x_1(t) x_2(t)$$

$$x_1(t) = \frac{\sin t}{\pi t}$$

$$x_2(t) = \frac{\sin \frac{t}{2}}{\pi t}$$

$$X = \frac{1}{2} X_1 * X_2$$



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Linear Constant-coefficient ODEs

Frequency response of LTI system described by

$$\sum_{k=0}^{N} a_k \frac{d^k y}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x}{dt^k}$$

Method 1. Use eigenfunction property

$$x(t) = e^{j\omega t} \implies y(t) = H(j\omega) e^{j\omega t}$$

Substitution into ODE yields

$$\sum_{k=0}^{N} a_k H(j\omega)(j\omega)^k e^{j\omega t} = \sum_{k=0}^{M} b_k (j\omega)^k e^{j\omega t}$$

$$H(j\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Linear Constant-coefficient ODEs

Method 2. Take Fourier transform of both sides

$$\mathcal{F}\left\{\sum_{k=0}^{N} a_k \frac{d^k y}{dt^k}\right\} = \mathcal{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k x}{dt^k}\right\}$$

By linearity and differentiation property

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

By convolution property

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Frequency response is rational function of $j\omega$

Example

$$y'' + 4y' + 3y = x' + 2x$$

Frequency response

$$H(j\omega) = \frac{j\omega + 2}{(j\omega)^2 + 4(j\omega) + 3}$$

By partial fraction expansion

$$H(j\omega) = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)} = \frac{1/2}{j\omega + 1} + \frac{1/2}{j\omega + 3}$$

Take inverse Fourier transform to find impulse response

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

Example

$$y'' + 4y' + 3y = x' + 2x$$

Find zero-state response to $x(t) = e^{-t}u(t)$

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)} \cdot \frac{1}{j\omega + 1} = \frac{j\omega + 2}{(j\omega + 1)^2(j\omega + 3)}$$

By partial fraction expansion

$$Y(j\omega) = \frac{1/4}{j\omega + 1} + \frac{1/2}{(j\omega + 1)^2} - \frac{1/4}{j\omega + 3}$$

Take inverse Fourier transform to find zero-state response

$$y(t) = \left(\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-3t}\right)u(t)$$

Inverse Fourier Transform of $\frac{1}{(j\omega+a)^n}$

Recall

$$e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+j\omega}$$

Note

$$j^{n-1}\frac{d^{n-1}}{d\omega^{n-1}}\left(\frac{1}{a+j\omega}\right) = \left(j\frac{d}{d\omega}\right)^{n-1}\left(\frac{1}{a+j\omega}\right) = \frac{(n-1)!}{(a+j\omega)^n}$$

Recall frequency differentiation property

$$tx(t) \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}X(j\omega)$$

Applying it $n-1$ times yields

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{(a+j\omega)^n}$$

Partial Fraction Expansion

Rational function

$$R(s) = \frac{P(s)}{Q(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Proper rational function if $M < N$

Improper rational function can be rewritten as sum of polynomial and proper rational function by long division

$$R(s) = P_0(s) + \frac{P_1(s)}{Q(s)}$$

where $\deg P_1 < \deg Q$

Example.

$$\frac{s^3 + 5s^2 + 8s + 5}{s^2 + 4s + 3} = s + 1 + \frac{s + 2}{s^2 + 4s + 3}$$

Partial Fraction Expansion

Example (long division).

$$\frac{s^3 + 5s^2 + 8s + 5}{s^2 + 4s + 3} = s + 1 + \frac{s + 2}{s^2 + 4s + 3}$$

$$\begin{array}{r} s + 1 \\ s^2 + 4s + 3 \overline{\smash{\big)}\, s^3 + 5s^2 + 8s + 5} \\ \underline{- s^3 - 4s^2 - 3s \phantom{{}+5}} \\ s^2 + 5s + 5 \\ \underline{- s^2 - 4s - 3} \\ s + 2 \end{array}$$

Partial Fraction Expansion

Consider proper rational function $R(s) = P(s)/Q(s)$

Denominator $Q(s)$ has r distinct roots $a_1, \ldots, a_r \in \mathbb{C}$ and factorization

$$Q(s) = \prod_{i=1}^{r} (s - a_i)^{N_i}$$

$R(s)$ has following partial fraction expansion

$$R(s) = \sum_{i=1}^{r} \sum_{k_i=1}^{N_i} \frac{A_{i,k_i}}{(s - a_i)^{k_i}}$$

Find coefficients by

- reducing to common denominator
- comparing coefficients of numerators

Partial Fraction Expansion

Example.

$$R(s) = \frac{s+2}{(s+1)^2(s+3)} = \frac{A_{1,1}}{s+1} + \frac{A_{1,2}}{(s+1)^2} + \frac{A_{2,1}}{s+3}$$

RHS is

$$\frac{A_{1,1}(s+1)(s+3) + A_{1,2}(s+3) + A_{2,1}(s+1)^2}{(s+1)^2(s+3)}$$
$$= \frac{(A_{1,1}+A_{2,1})s^2 + (4A_{1,1}+A_{1,2}+2A_{2,1})s + (3A_{1,1}+3A_{1,2}+A_{2,1})}{(s+1)^2(s+3)}$$

$$\begin{cases} A_{1,1} + A_{2,1} = 0 \\ 4A_{1,1} + A_{1,2} + 2A_{2,1} = 1 \\ 3A_{1,1} + 3A_{1,2} + A_{2,1} = 2 \end{cases} \Longrightarrow \begin{cases} A_{1,1} = 1/4 \\ A_{1,2} = 1/2 \\ A_{2,1} = -1/4 \end{cases}$$

Partial Fraction Expansion

$$R(s) = \sum_{i=1}^{r} \sum_{k_i=1}^{N_i} \frac{A_{i,k_i}}{(s-a_i)^{k_i}}$$

Multiply both sides by $(s-a_j)^{N_j}$,

$$(s-a_j)^{N_j} R(s) = \sum_{k_j=1}^{N_j} A_{j,k_j} (s-a_j)^{N_j-k_j} + (s-a_j)^{N_j} R_j(s)$$

where $R_j(s) = \sum_{i \neq j} \sum_{k_i=1}^{N_i} \frac{A_{i,k_i}}{(s-a_i)^{k_i}}$. For $k < N_j$,

$$\frac{d^k}{ds^k}\left[(s-a_j)^{N_j} R(s)\right] = \sum_{k_j=0}^{N_j-k} A_{j,k_j} \frac{(N_j-k_j)!}{(N_j-k_j-k)!} (s-a_j)^{N_j-k_j-k}$$

$$+ \sum_{\ell=0}^{k} \binom{k}{\ell} \frac{N_j!}{(N_j-\ell)!} (s-a_j)^{N_j-\ell} R_j^{(\ell)}(s)$$

Partial Fraction Expansion

$$\frac{d^k}{ds^k}\left[(s-a_j)^{N_j}R(s)\right] = \sum_{k_j=0}^{N_j-k} A_{j,k_j}\frac{(N_j-k_j)!}{(N_j-k_j-k)!}(s-a_j)^{N_j-k_j-k}$$
$$+\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{N_j!}{(N_j-\ell)!}(s-a_j)^{N_j-\ell}R_j^{(\ell)}(s)$$

Evaluating at $s=a_j$,

$$\left[\frac{d^k}{ds^k}\left[(s-a_j)^{N_j}R(s)\right]\right]\Bigg|_{s=a_j} = A_{j,N_j-k}k!$$

Replacing k by N_j-k,

$$A_{j,k} = \frac{1}{(N_j-k)!}\left[\frac{d^{N_j-k}}{ds^{N_j-k}}\left[(s-a_j)^{N_j}R(s)\right]\right]\Bigg|_{s=a_j}$$

Partial Fraction Expansion

Example.

$$R(s) = \frac{s+2}{(s+1)^2(s+3)} = \frac{A_{1,1}}{s+1} + \frac{A_{1,2}}{(s+1)^2} + \frac{A_{2,1}}{s+3}$$

To find $A_{2,1}$

1. Multiply both sides by $s+3$,

$$(s+3)R(s) = \frac{s+2}{(s+1)^2} = \frac{A_{1,1}(s+3)}{s+1} + \frac{A_{1,2}(s+3)}{(s+1)^2} + A_{2,1}$$

2. Evaluate at $s = -3$,

$$A_{2,1} = \left.\frac{s+2}{(s+1)^2}\right|_{s=-3} = -\frac{1}{4}$$

Partial Fraction Expansion

Example (cont'd).

$$R(s) = \frac{s+2}{(s+1)^2(s+3)} = \frac{A_{1,1}}{s+1} + \frac{A_{1,2}}{(s+1)^2} + \frac{A_{2,1}}{s+3}$$

To find $A_{1,2}$

1. Multiply both sides by $(s+1)^2$,

$$(s+1)^2 R(s) = \frac{s+2}{s+3} = A_{1,1}(s+1) + \ A_{1,2} + \frac{A_{2,1}(s+1)^2}{s+3}$$

2. Evaluate at $s=-1$,

$$A_{1,2} = \left.\frac{s+2}{s+3}\right|_{s=-1} = \frac{1}{2}$$

Partial Fraction Expansion

Example (cont'd).

$$R(s) = \frac{s+2}{(s+1)^2(s+3)} = \frac{A_{1,1}}{s+1} + \frac{A_{1,2}}{(s+1)^2} + \frac{A_{2,1}}{s+3}$$

To find $A_{1,1}$

1. Multiply both sides by $(s+1)^2$,

$$(s+1)^2 R(s) = \frac{s+2}{s+3} = A_{1,1}(s+1) + \ A_{1,2} + \frac{A_{2,1}(s+1)^2}{s+3}$$

2. Take first derivative w.r.t. s,

$$\frac{d}{ds}\left(\frac{s+2}{s+3}\right) = \frac{1}{(s+3)^2} = A_{1,1} + \frac{d}{ds}\left(\frac{A_{2,1}(s+1)^2}{s+3}\right)$$

3. Evaluate at $s = -1$,

$$A_{1,1} = \left.\frac{1}{(s+3)^2}\right|_{s=-1} = \frac{1}{4}$$

Partial Fraction Expansion

Example.

$$R(s) = \frac{1}{(s+1)^3(s+3)} = \frac{A_{1,1}}{s+1} + \frac{A_{1,2}}{(s+1)^2} + \frac{A_{1,3}}{(s+1)^3} + \frac{A_{2,1}}{s+3}$$

1. $A_{2,1}$

$$(s+3)R(s) = \frac{1}{(s+1)^3} = \frac{A_{1,1}(s+3)}{s+1} + \frac{A_{1,2}(s+3)}{(s+1)^2} + \frac{A_{1,3}(s+3)}{(s+1)^3} + A_{2,1}$$

$$A_{2,1} = \left.\frac{1}{(s+1)^3}\right|_{s=-3} = -\frac{1}{8}$$

2. $A_{1,3}$

$$(s+1)^3R(s) = \frac{1}{s+3} = A_{1,1}(s+1)^2 + A_{1,2}(s+1) + A_{1,3} + \frac{A_{2,1}(s+1)^3}{s+3}$$

$$A_{1,3} = \left.\frac{1}{s+3}\right|_{s=-1} = \frac{1}{2}$$

Partial Fraction Expansion

Example (cont'd).

$$R(s) = \frac{1}{(s+1)^3(s+3)} = \frac{A_{1,1}}{s+1} + \frac{A_{1,2}}{(s+1)^2} + \frac{A_{1,3}}{(s+1)^3} + \frac{A_{2,1}}{s+3}$$

3. $A_{1,2}$

$$(s+1)^3 R(s) = \frac{1}{s+3} = A_{1,1}(s+1)^2 + A_{1,2}(s+1) + A_{1,3} + \frac{A_{2,1}(s+1)^3}{s+3}$$

$$A_{1,2} = \frac{d}{ds}\left(\frac{1}{s+3}\right)\Big|_{s=-1} = -\frac{1}{(s+3)^2}\Big|_{s=-1} = -\frac{1}{4}$$

4. $A_{1,1}$

$$2A_{1,1} = \frac{d^2}{ds^2}\left(\frac{1}{s+3}\right)\Big|_{s=-1} = \frac{2}{(s+3)^3}\Big|_{s=-1} = \frac{1}{8} \implies A_{1,1} = \frac{1}{16}$$