EI331 Signals and Systems

Lecture 17

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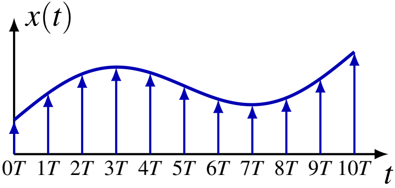
Contents

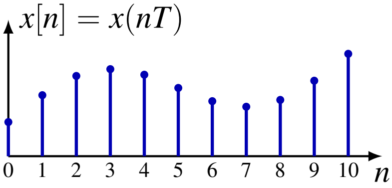
Sampling CT Signals

Sampling converts CT signals to DT signals

CT signal



DT signal



T = sampling period, uniform sampling most common

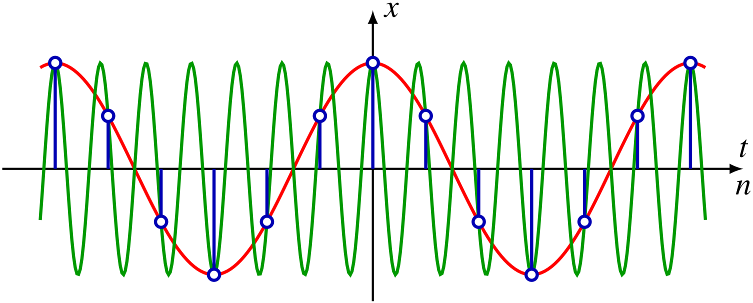
Allows use of digital electronics to process, record, transmit, store, and retrieve CT signals

- MP3, digital camera, printer

Sampling CT Signals

Sampling loses information, different signals may have same samples

- $x_1(t) = \cos(\frac{\pi}{3}t)$, $x_2(t) = \cos(\frac{7\pi}{3}t)$, **different**
- $x_1[n] = \cos(\frac{\pi}{3}n) = x_2[n] = \cos(\frac{7\pi}{3}n)$, **identical**

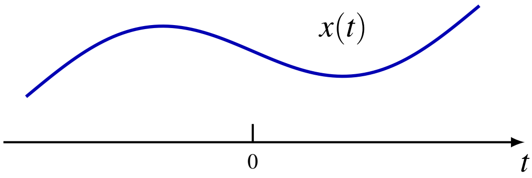


Under what conditions can we recover signal from samples?

Impulse-train Sampling

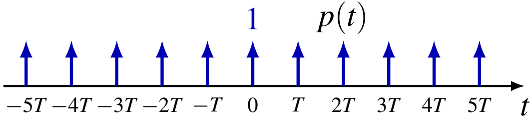
Time domain

CT signal $x(t)$



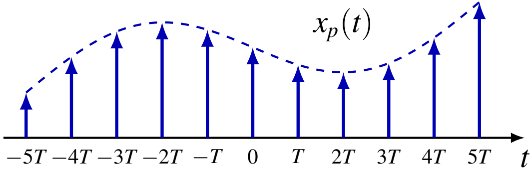
impulse train

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



$$x_p(t) = x(t)p(t)$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$



Impulse-train Sampling

Frequency domain

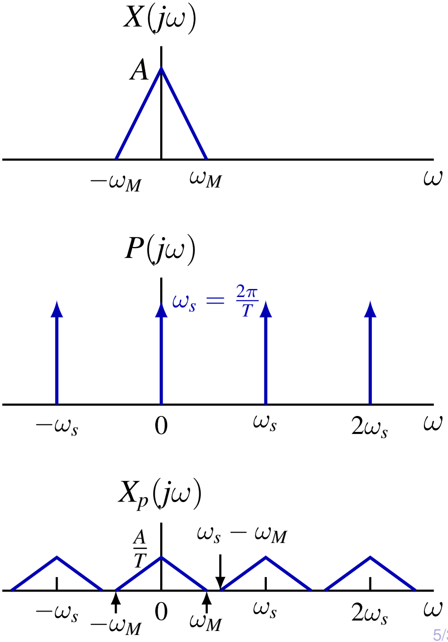
CT signal $X(j\omega)$

impulse train

$$P(j\omega) = \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

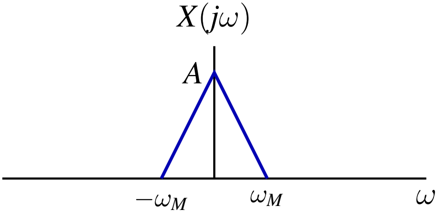
$$X_p(j\omega) = \frac{1}{2\pi}(X * P)(\omega)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$



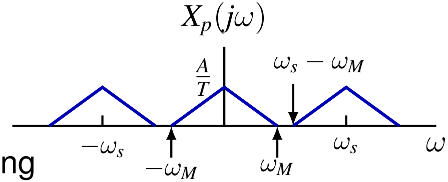
Impulse-train Sampling

Frequency domain

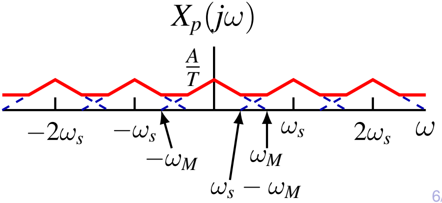


band-limited CT signal
$X(j\omega) = 0$ for $|\omega| > \omega_M$

Sampling frequency $\omega_s = \frac{2\pi}{T}$



Case 1: $\omega_s > 2\omega_M$
no overlap between replicas
can recover X by lowpass filtering



Case 2: $\omega_s < 2\omega_M$
replicas **overlap**
cannot recover X

Sampling Theorem

Band-limited CT signal $x(t)$ whose spectrum $X(j\omega) = 0$ for $|\omega| > \omega_M$ is uniquely determined by its samples $x(nT), n \in \mathbb{Z}$ if

$$\omega_s \triangleq \frac{2\pi}{T} > 2\omega_M, \quad 2\omega_M \text{ called Nyquist rate}$$

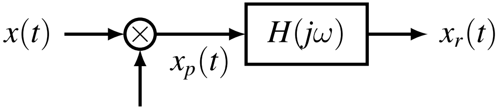
Given $\{x(nT) : n \in \mathbb{Z}\}$, $x(t)$ can be reconstructed as follows

1. construct $x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$
2. send x_p through lowpass filter with gain T and cutoff frequency $\omega_c \in (\omega_M, \omega_s - \omega_M)$, i.e.

$$H(j\omega) = T[u(\omega + \omega_c) - u(\omega - \omega_c)]$$

3. filter output $x_r(t)$ with $X_r(j\omega) = X_p(j\omega)H(j\omega)$ is same as $x(t)$

Reconstruction in Frequency Domain

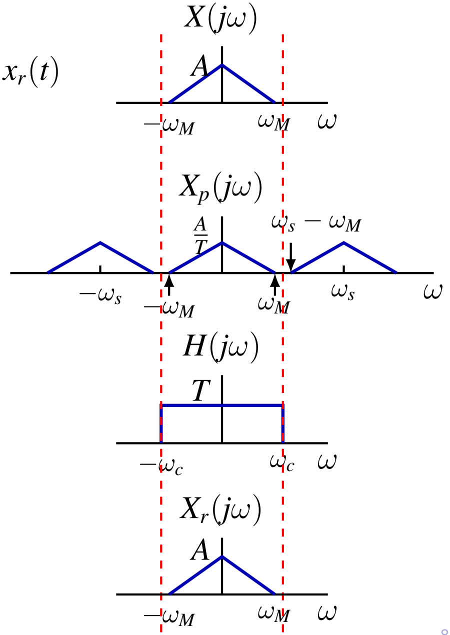


$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

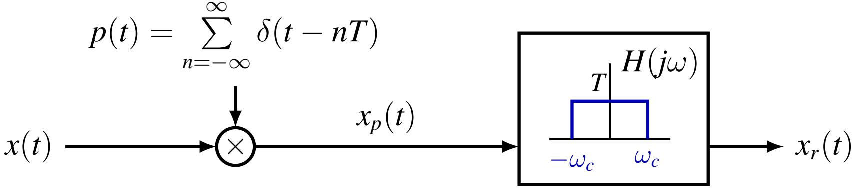
- Nyquist frequency ω_M
- Nyquist rate $2\omega_M$

Lowpass filter

- gain T
- cutoff frequency
  $\omega_M < \omega_c < \omega_s - \omega_M$



Reconstruction in Time Domain



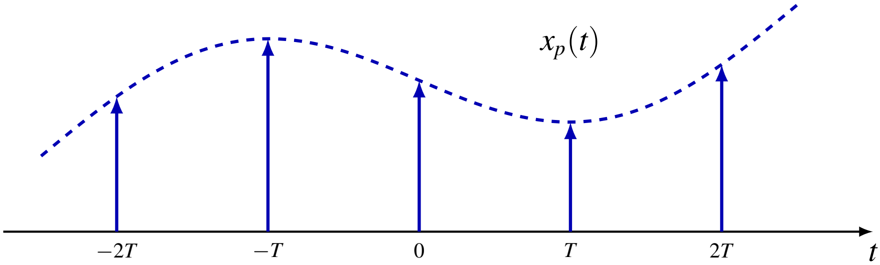
Impulse response of lowpass filter

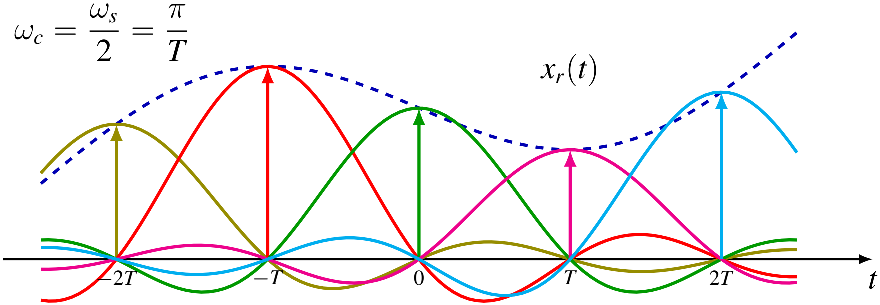
$$h(t) = \frac{T \sin(\omega_c t)}{\pi t}$$

x recovered by band-limited interpolation using sinc function

$$x(t) = x_r(t) = (x_p * h)(t) = \sum_{n=-\infty}^{\infty} x(nT) \frac{T \sin(\omega_c (t - nT))}{\pi (t - nT)}$$

Reconstruction in Time Domain





Reconstruction in Time Domain

Setting $\omega_c = \frac{\omega_s}{2}$ yields Whittaker-Shannon interpolation formula

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}\left(\frac{t-nT}{T}\right), \quad \text{where } \mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

Since

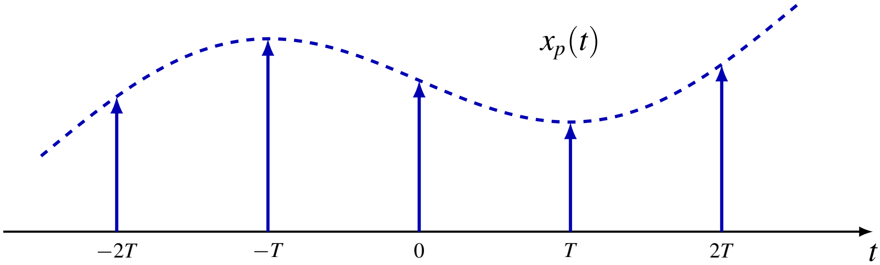
$$\mathrm{sinc}\left(\frac{t-nT}{T}\right) \overset{\mathcal{F}}{\longleftrightarrow} Te^{-jnT\omega}[u(\omega+\frac{\pi}{T}) - u(\omega-\frac{\pi}{T})]$$

Parseval's identity (or multiplication property) implies

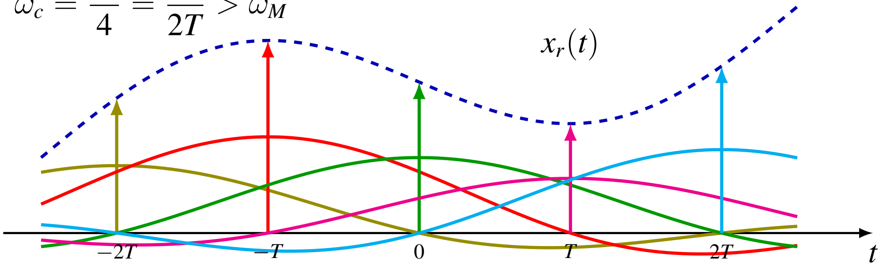
$$\int_{\mathbb{R}} \mathrm{sinc}\left(\frac{t-nT}{T}\right)\mathrm{sinc}\left(\frac{t-mT}{T}\right)dt = \frac{T^2}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} e^{j(m-n)T\omega}d\omega = T\delta[n-m]$$

Whittaker-Shannon formula is orthogonal expansion

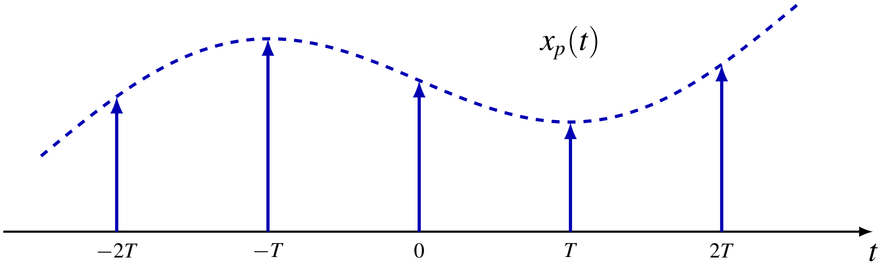
Reconstruction in Time Domain



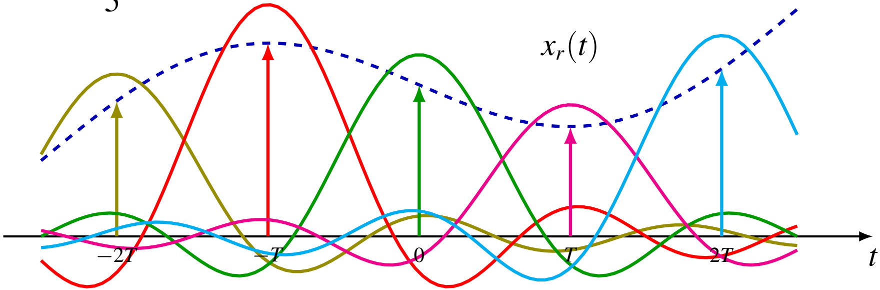
$$\omega_c = \frac{\omega_s}{4} = \frac{\pi}{2T} > \omega_M$$



Reconstruction in Time Domain



$$\omega_c = \frac{3\omega_s}{5} < \omega_s - \omega_M$$



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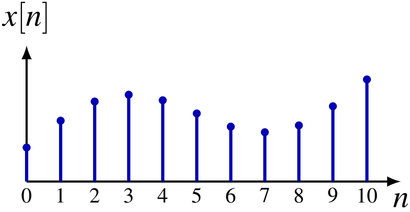
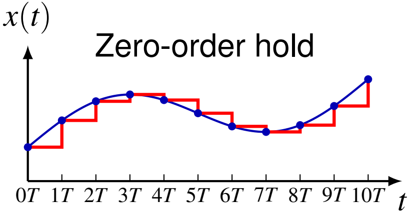
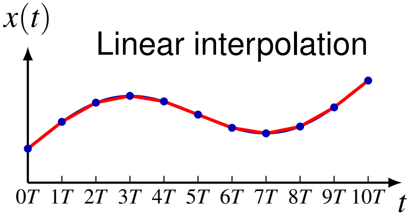
Other Interpolation Methods



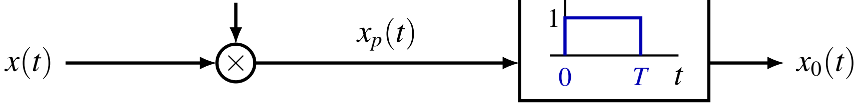
T = sampling period





Zero-order Hold

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

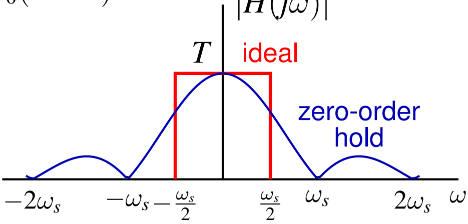


Reconstructed signal

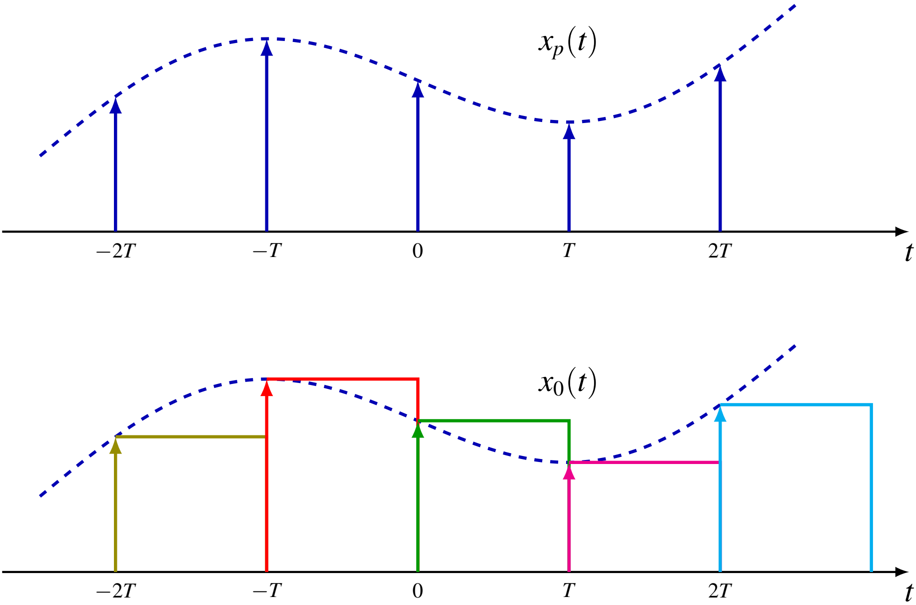
$$x_0(t) = (x_p * h_0)(t) = \sum_{n=-\infty}^{\infty} x(nT)h_0(t - nT)$$

Zero-order hold filter

$$H_0(j\omega) = e^{-j\omega T/2} \frac{2\sin(\omega T/2)}{\omega}$$

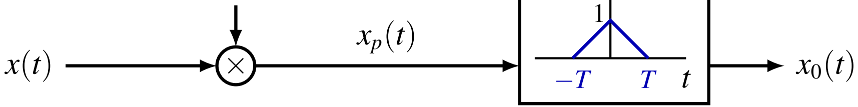


Zero-order Hold



Linear Interpolation (First-order Hold)

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

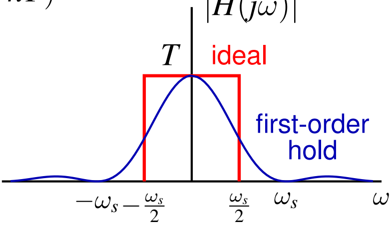


Reconstructed signal

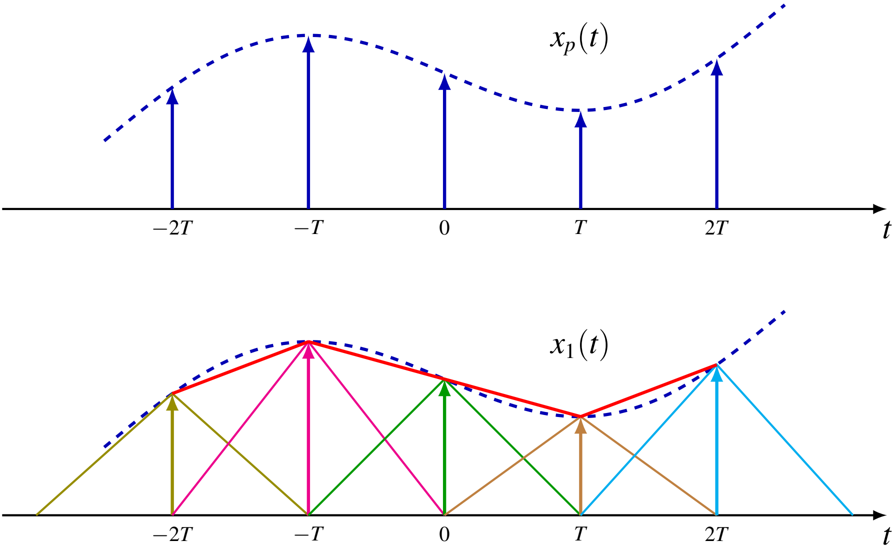
$$x_1(t) = (x_p * h_1)(t) = \sum_{n=-\infty}^{\infty} x(nT) h_1(t - nT)$$

First-order hold filter

$$H_1(j\omega) = \frac{1}{T} \left[\frac{\sin(\omega T/2)}{\omega/2} \right]^2$$



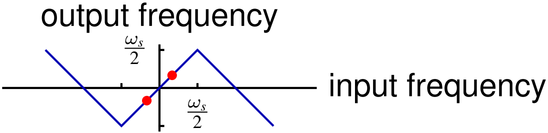
Linear Interpolation



Contents

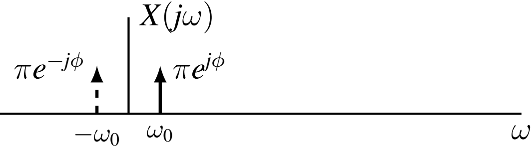
Aliasing

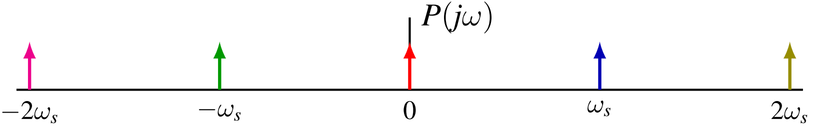
Aliasing wraps frequencies



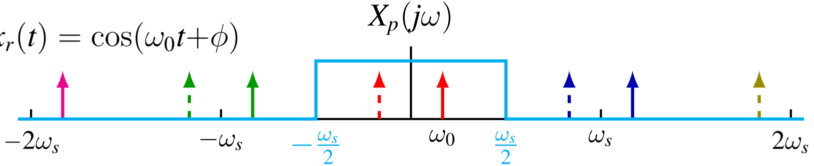
$x(t) = \cos(\omega_0 t + \phi)$

$\omega_0 < \frac{1}{2}\omega_s$



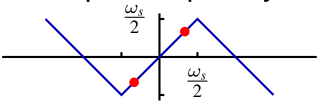


$x_r(t) = \cos(\omega_0 t + \phi)$



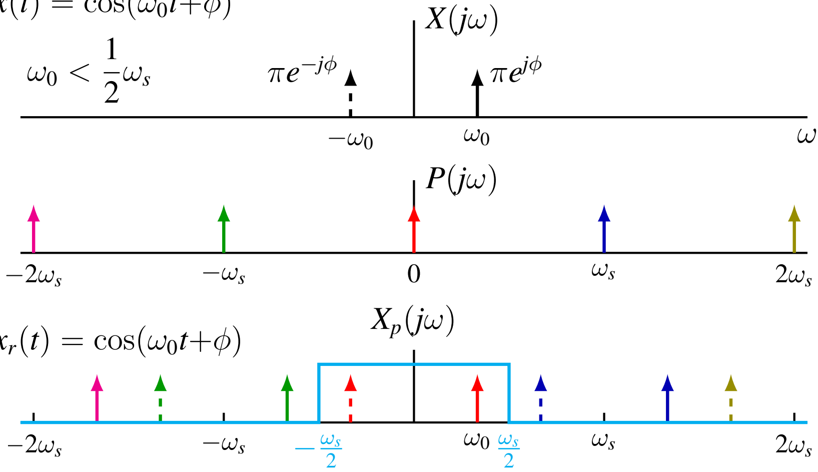
Aliasing

Aliasing wraps frequencies



$x(t) = \cos(\omega_0 t + \phi)$

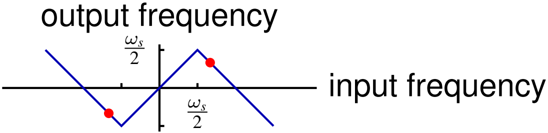
$\omega_0 < \frac{1}{2}\omega_s$



$x_r(t) = \cos(\omega_0 t + \phi)$

Aliasing

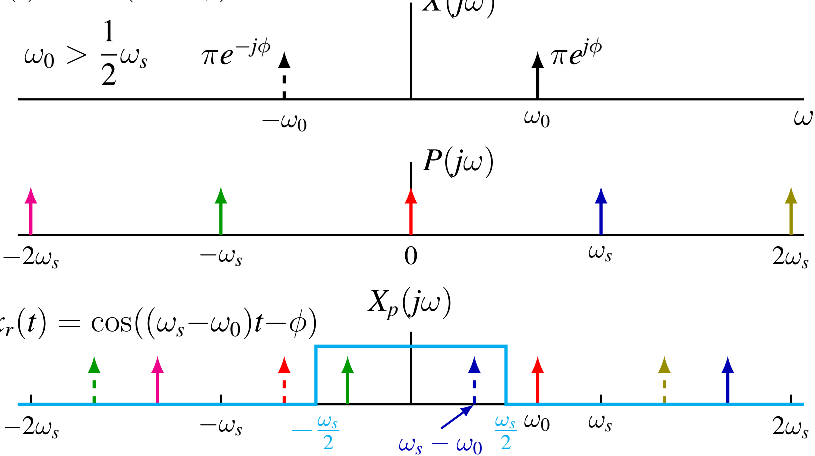
Aliasing wraps frequencies



$x(t) = \cos(\omega_0 t + \phi)$

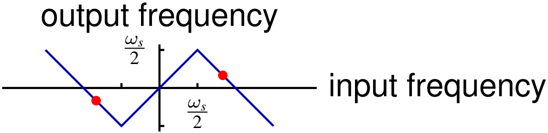
$\omega_0 > \frac{1}{2}\omega_s$

$x_r(t) = \cos((\omega_s - \omega_0)t - \phi)$

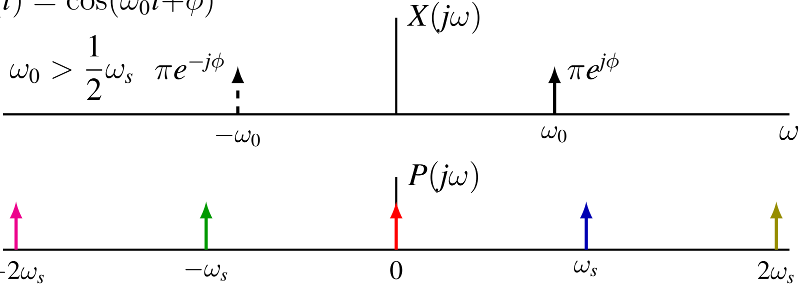


Aliasing

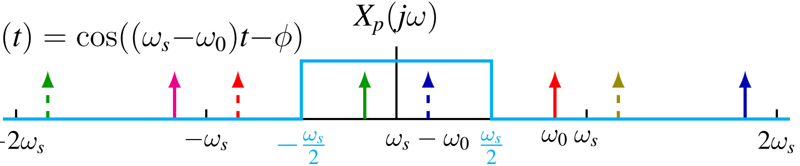
Aliasing wraps frequencies



$x(t) = \cos(\omega_0 t + \phi)$

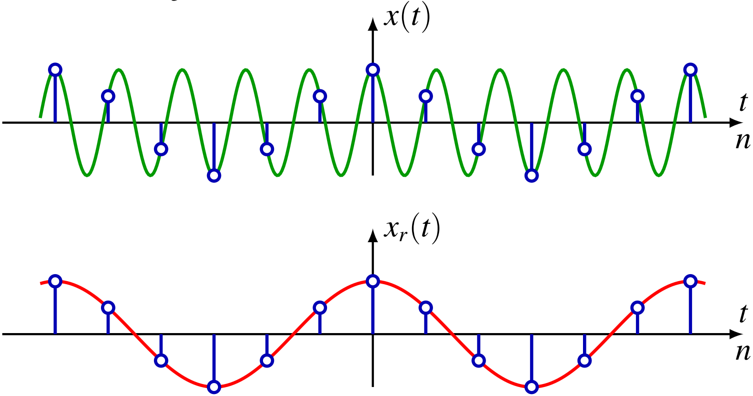


$x_r(t) = \cos((\omega_s - \omega_0)t - \phi)$



Aliasing

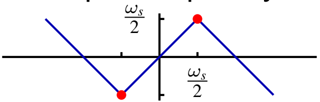
- $x(t) = \cos(\frac{5\pi}{3}t)$, $x[n] = \cos(\frac{5\pi}{3}n) = \cos(\frac{\pi}{3}n)$
- $x_r(t) = \cos(\frac{\pi}{3}t)$



Example. In movies, wheels often appear to rotate more slowly than they actually do and even in wrong direction

Aliasing

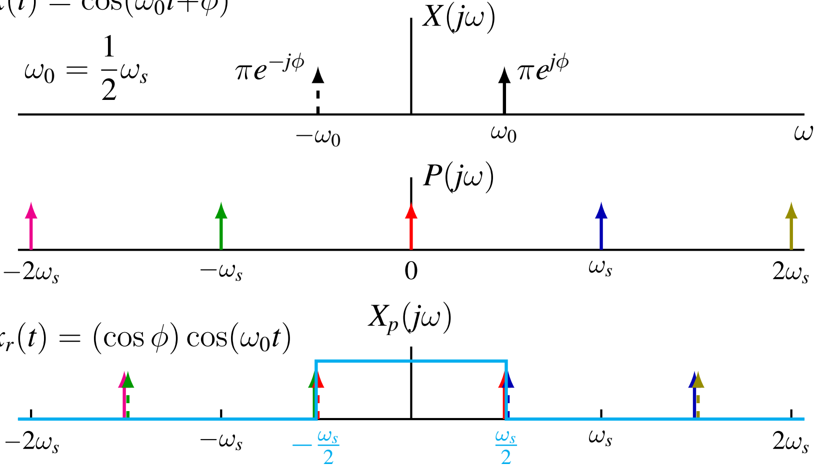
Aliasing wraps frequencies



$x(t) = \cos(\omega_0 t + \phi)$

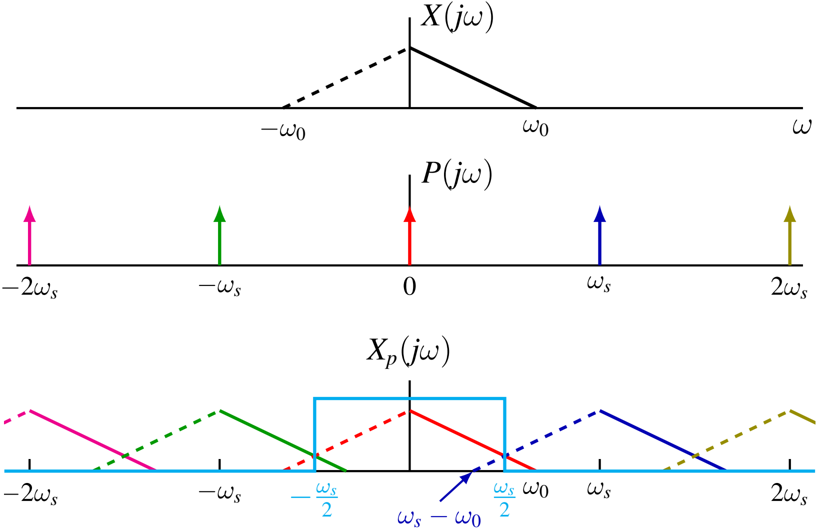
$\omega_0 = \frac{1}{2}\omega_s$

$x_r(t) = (\cos\phi)\cos(\omega_0 t)$



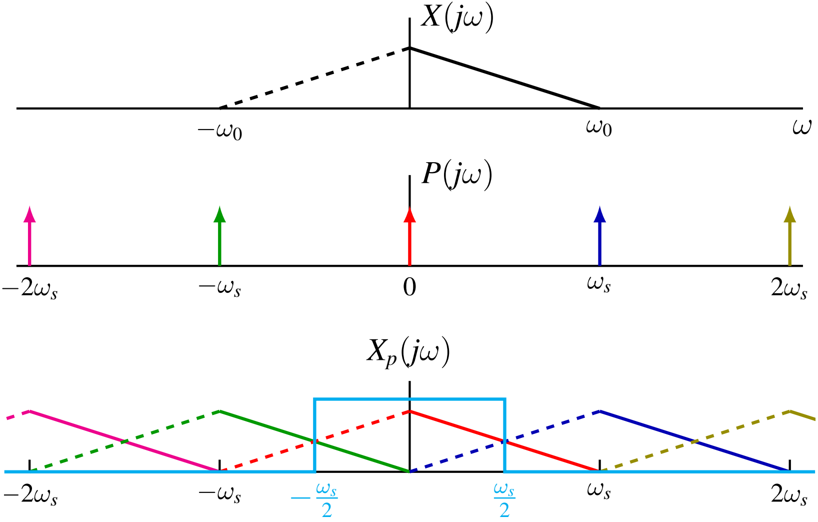
Aliasing

Aliasing for more complex signals also wraps frequencies



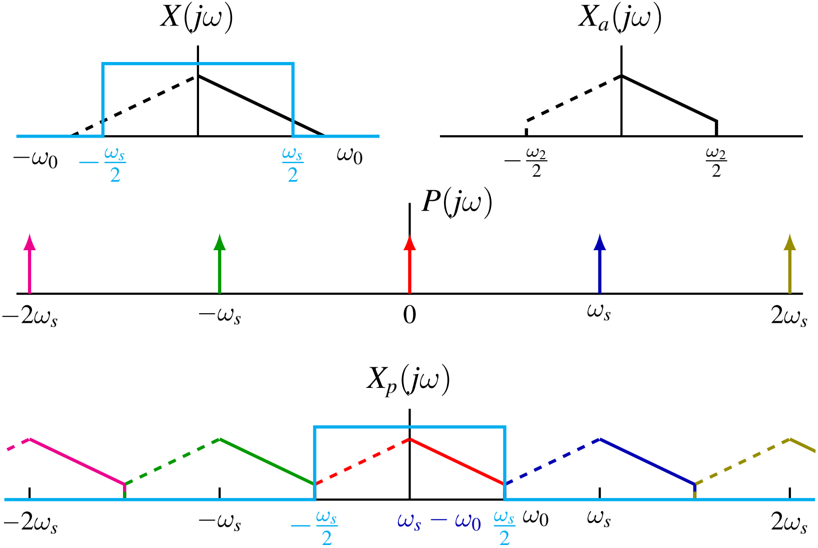
Aliasing

Aliasing increases as sampling rate decreases



Anti-aliasing Filter

Filter out frequencies above $\frac{\omega_s}{2}$ before sampling



Anti-aliasing Filter

Filter out frequencies above $\frac{\omega_s}{2}$ before sampling