EI331 Signals and Systems

Lecture 18

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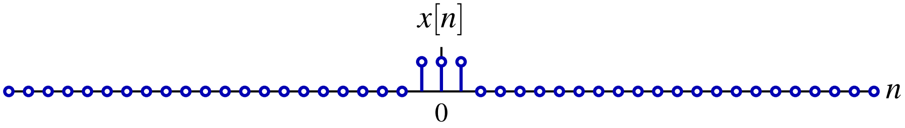
Contents

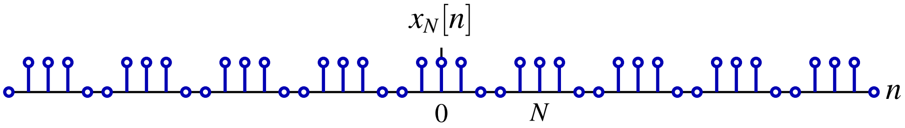
DT Fourier Transform

Let x be aperiodic DT signal with $\operatorname{supp} x \subset [N_1, N_2]$



Define periodic extension with period $N > N_2 - N_1 + 1$,

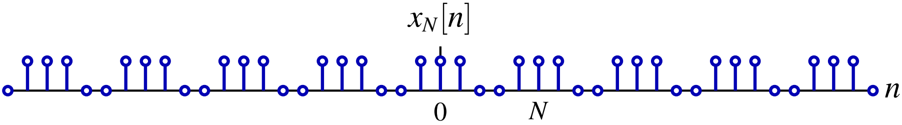
$$x_N[n] = \sum_{k=-\infty}^{\infty} x[n + kN]$$



x is "periodic" with infinite period, i.e. $x[n] = \lim_{N\to\infty} x_N[n]$

DT Fourier Transform

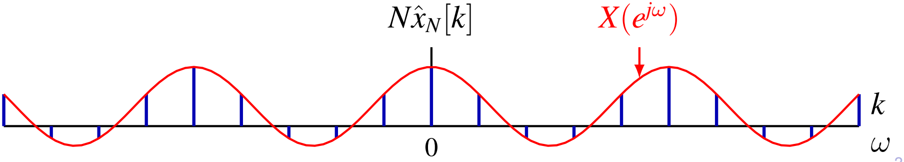
Expand x_N into DT Fourier series



$$\hat{x}_N[k] = \langle x_N, e^{j\frac{2\pi}{N}k} \rangle = \frac{1}{N} \sum_{n=N_1}^{N_2} x[n] e^{-j\frac{2\pi}{N}k} = \frac{1}{N} X(e^{jk\omega_0})$$

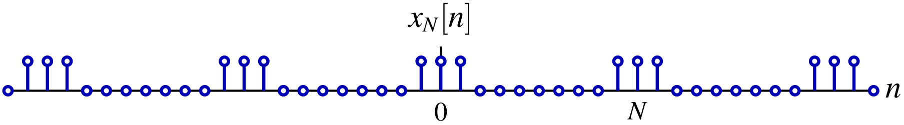
where

$$X(e^{j\omega}) = \sum_{n=N_1}^{N_2} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}, \quad \omega_0 = \frac{2\pi}{N}$$



DT Fourier Transform

As N increases, discrete frequencies sampled more densely



$$\hat{x}_N[k] = \langle x_N, e^{j\frac{2\pi}{N}k} \rangle = \frac{1}{N} \sum_{n=N_1}^{N_2} x[n] e^{-j\frac{2\pi}{N}k} = \frac{1}{N} X(e^{jk\omega_0})$$

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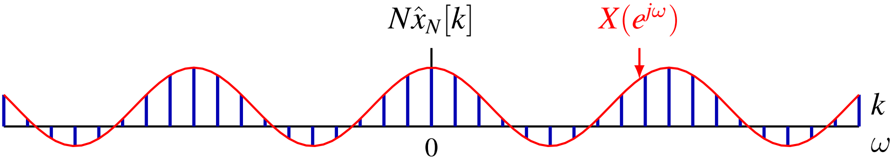
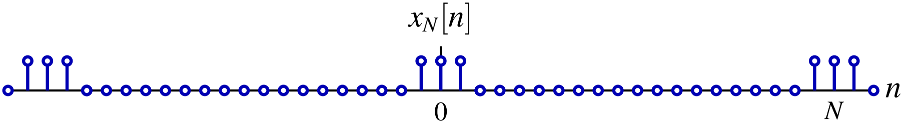


DT Fourier Transform

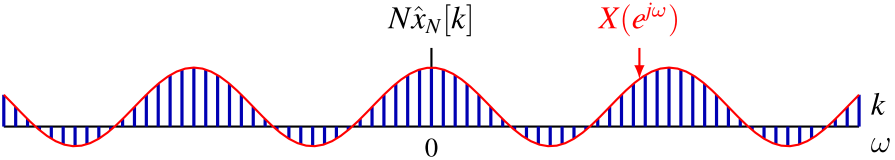
As $N \to \infty$, $N\hat{x}[k]$ approaches envelope $X(e^{j\omega})$



$$\hat{x}_N[k] = \langle x_N, e^{j\frac{2\pi}{N}k} \rangle = \frac{1}{N} \sum_{n=N_1}^{N_2} x[n] e^{-j\frac{2\pi}{N}k} = \frac{1}{N} X(e^{jk\omega_0})$$

where

$$X(e^{j\omega}) = \sum_{n=N_1}^{N_2} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}, \quad \omega_0 = \frac{2\pi}{N}$$



DT Fourier Transform

Synthesis equation for DT Fourier series,

$$x_N[n] = \sum_{k\in[N]} \hat{x}[k]e^{jk\omega_0 n} = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{jk\omega_0})e^{jk\omega_0 n}$$

Let $\omega_k = k\omega_0$, $\Delta\omega = \omega_k - \omega_{k-1} = \omega_0$,

$$x_N[n] = \frac{1}{2\pi}\sum_{k=0}^{N-1} X(e^{j\omega_k})e^{j\omega_k n}\Delta\omega$$

As $N \to \infty$, $\Delta\omega \to 0$, above Riemann sum becomes integral,

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Can integrate over any period of length 2π

DT Fourier Transform Pair

DT Fourier transform (analysis equation)

$$\boxed{X(e^{j\omega}) = \mathcal{F}(x)(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}}$$

- $X(e^{j\omega})$ called spectrum of $x[n]$, **periodic** with period 2π

DT Inverse Fourier transform (synthesis equation)

$$\boxed{x[n] = \mathcal{F}^{-1}(X)[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega}$$

- Complex exponential at frequency ω has weight $X(e^{j\omega})\frac{d\omega}{2\pi}$
- Integrate over **single period**, i.e. only use **distinct** $e^{j\omega n}$

Duality between DTFT and CTFS

DTFT pair

- discrete time
- continuous frequency

analysis equation

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

synthesis equation

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

CTFS pair

- continuous time
- discrete frequency

analysis equation

$$\hat{x}[k] = \frac{1}{T}\int_{T} x(t)e^{-jk\frac{2\pi}{T}t}dt$$

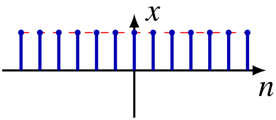
synthesis equation

$$x(t) = \sum_{k=-\infty}^{\infty} \hat{x}[k]e^{jk\frac{2\pi}{T}t}$$

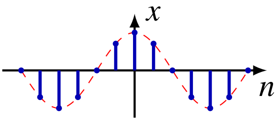
continuous variable periodic functions $\underset{\text{DTFT}}{\overset{\text{CTFS}}{\rightleftarrows}}$ doubly infinite sequences

High vs. Low Frequencies for DT Signals

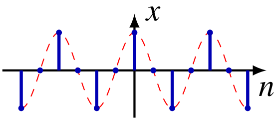
High frequencies around $(2k+1)\pi$, low frequencies around $2k\pi$



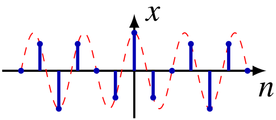
$\phi_N^0[n] = \cos(0 \cdot n) = 1$



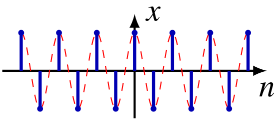
$\phi_N^1[n] = \cos(\pi n/4)$



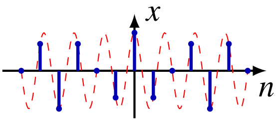
$\phi_N^2[n] = \cos(\pi n/2)$



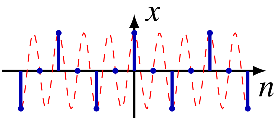
$\phi_N^3[n] = \cos(3\pi n/4)$



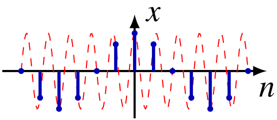
$\phi_N^4[n] = \cos(\pi n)$



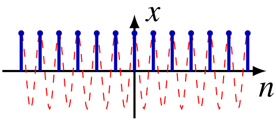
$\phi_N^5[n] = \cos(5\pi n/4)$



$\phi_N^6[n] = \cos(3\pi n/2)$



$\phi_N^7[n] = \cos(7\pi n/4)$

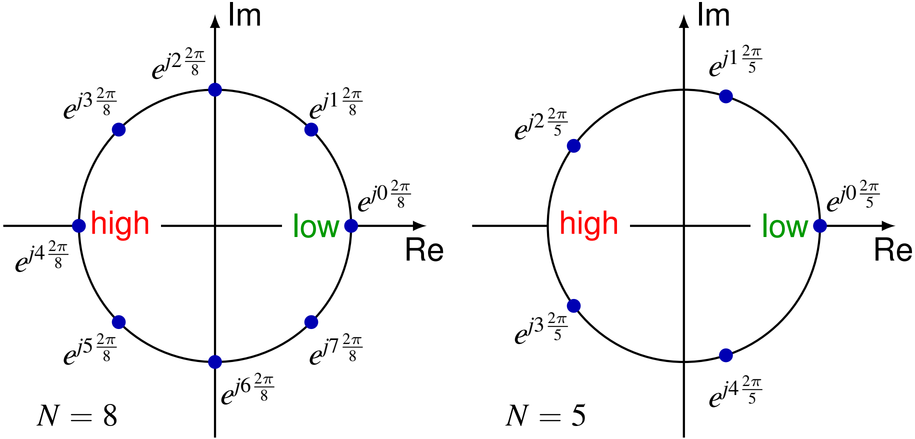


$\phi_N^8[n] = \cos(2\pi n) = 1$

High vs. Low Frequencies for DT Signals

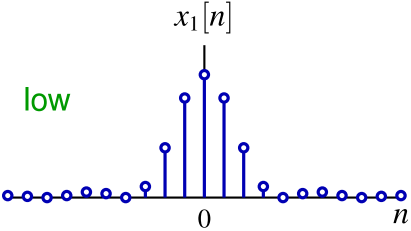
Discrete frequencies of N-periodic signals

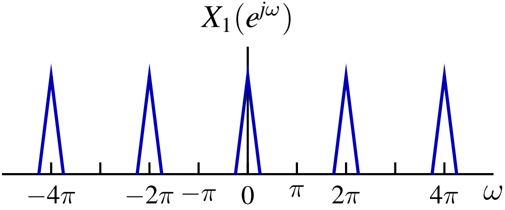
- evenly spaced points on unit circle
- low frequencies close to 1; high frequencies close to -1



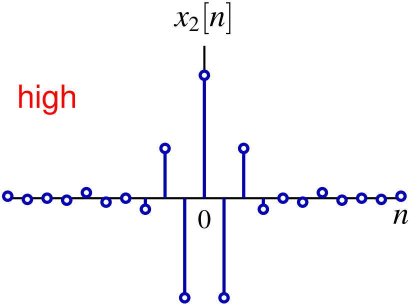
High vs. Low Frequencies for DT Signals

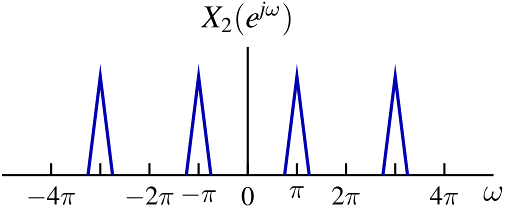
low





high

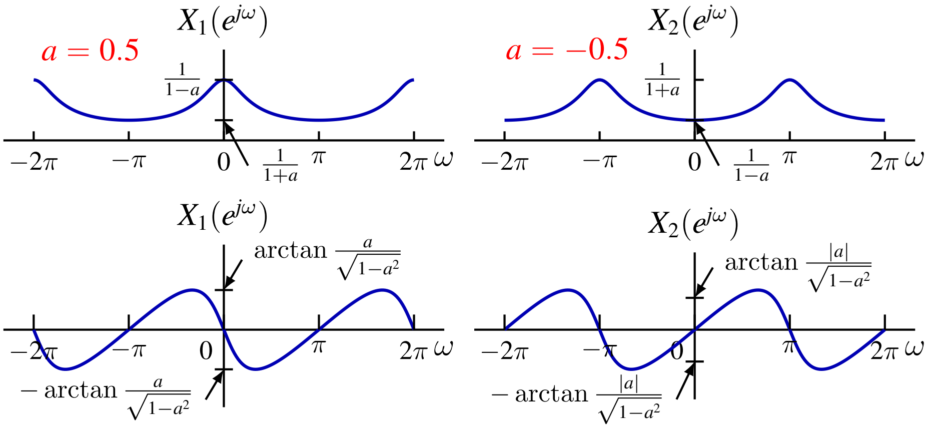




Example: One-sided Decaying Exponential

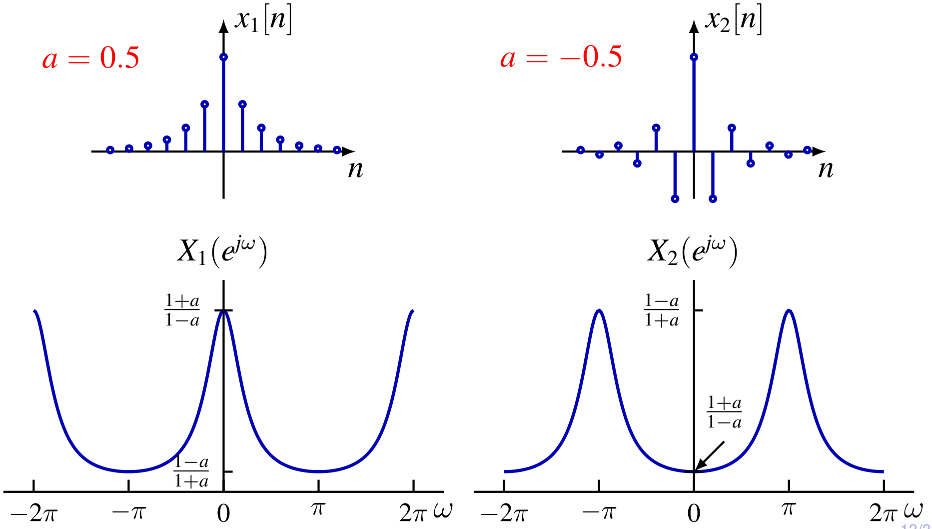
$$x[n] = a^n u[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}, \quad |a| < 1$$

$$|X(e^{j\omega})| = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}}, \quad \arg X(e^{j\omega}) = -\arctan\frac{a\sin\omega}{1 - a\cos\omega}$$



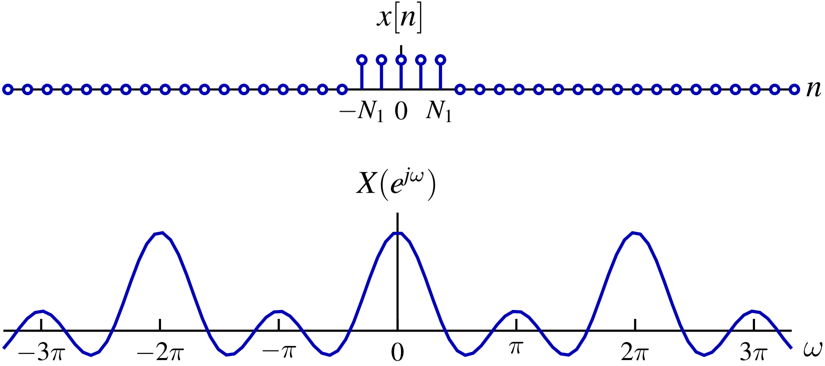
Example: Two-sided Decaying Exponential

$$x[n] = a^{|n|} \xleftrightarrow{\mathcal{F}} X(e^{j\omega}) = \frac{1-a^2}{1-2a\cos\omega + a^2}, \quad |a| < 1$$



Example: Rectangular Pulse

$$x[n] = u[n + N_1] - u[n - N_1 - 1] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) = \frac{\sin(\frac{2N_1+1}{2}\omega)}{\sin\frac{\omega}{2}}$$



Dirichlet kernel (periodic) is DT counterpart of sinc (aperiodic)

NB. $N_1 = 0 \implies x[n] = \delta[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) = 1$

Convergence of DTFT

Let

$$X_N(e^{j\omega}) = \sum_{n=-N}^{N} x[n]e^{-j\omega n}$$

Theorem. If $x \in \ell_1$, then X_N converges to X uniformly, i.e.

$$\lim_{N\to\infty} \|X - X_N\|_\infty = 0$$

Consequently, Fourier transform X is uniformly continuous.

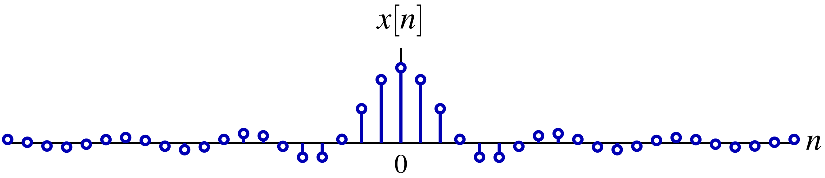
Theorem. If $x \in \ell_2$, then X_N converges to X in L_2, i.e.

$$\lim_{N\to\infty} \|X - X_N\|_2 = 0$$

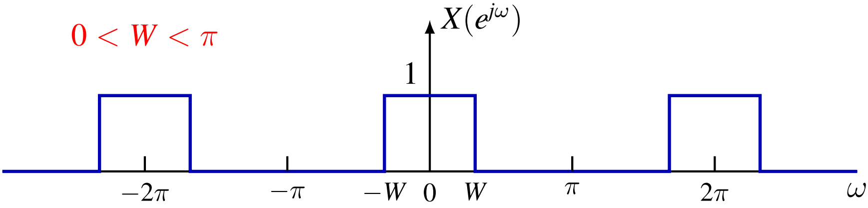
- CTFS: $L_2(T) \to \ell_2$ is isomorphism between Hilbert spaces
- DTFT: $\ell_2 \to L_2(2\pi)$ is also isomorphism

Example: Ideal Lowpass Filter

$$x[n] = \frac{\sin(Wn)}{\pi n} \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} [u(\omega+W+2k\pi) - u(\omega-W+2k\pi)]$$



$0 < W < \pi$



NB. $x \notin \ell_1$ and X discontinuous

Convergence as Generalized Function

Complex exponential $x[n] = e^{j\omega_0 n}$ not in ℓ_1 or ℓ_2.

(Shifted) Dirichlet kernel

$$X_N(e^{j\omega n}) = \frac{\sin(\frac{2N_1+1}{2}(\omega - \omega_0))}{\sin\frac{\omega-\omega_0}{2}} \to 2\pi \sum_{\ell=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\ell\pi)$$

in the sense of distribution (generalized function), i.e.

$$x[n] = e^{j\omega_0 n} \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) = 2\pi \sum_{\ell=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\ell\pi)$$

Formal verification using synthesis equation

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega = \int_{-\pi}^{\pi} \delta(\omega - \omega_0)e^{j\omega n}d\omega = e^{j\omega_0 n}$$

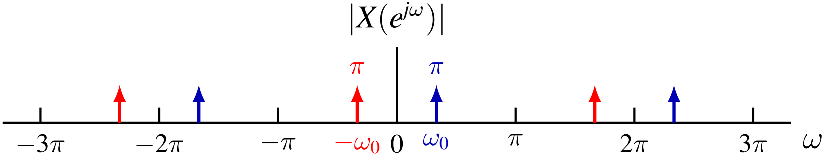
NB. Can also obtain by sampling $e^{j\omega_0 t}$ with $T = 1$, $\omega_s = 2\pi$

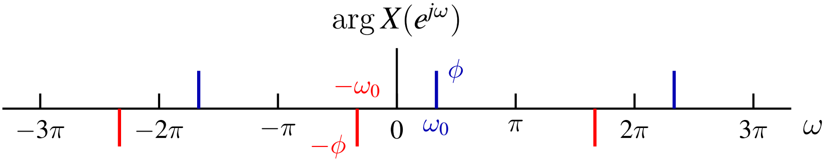
NB. $x[n]$ not necessarily periodic! DC corresponds to $\omega_0 = 0$.

Example: Sinusoids

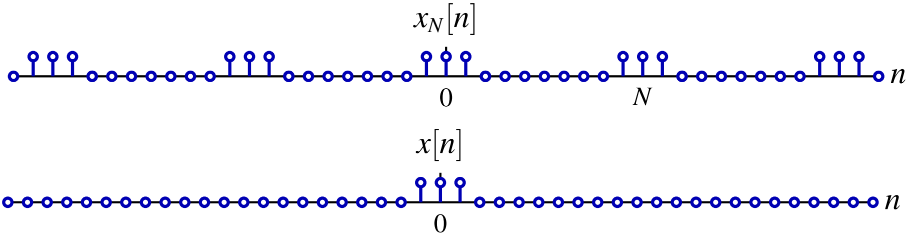
$$x[n] = \cos(\omega_0 n + \phi) = \frac{e^{j\phi}}{2} e^{j\omega_0 n} + \frac{e^{-j\phi}}{2} e^{-j\omega_0 n}$$

$$X(e^{j\omega}) = \sum_{\ell=-\infty}^{\infty} \pi e^{j\phi} \delta(\omega - \omega_0 - 2\ell\pi) + \pi e^{-j\phi} \delta(\omega + \omega_0 - 2\ell\pi)$$



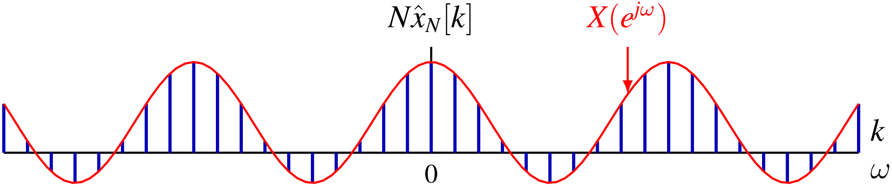


DTFT for Periodic Signals

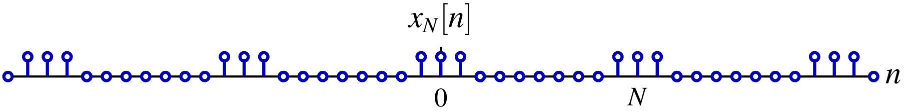


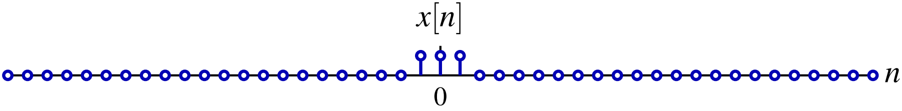
Recall DTFS coefficient $\hat{x}_N[k]$ is amplitude at frequency $k\omega_0$

$$\hat{x}_N[k] = \frac{1}{N} X(e^{jk\omega_0}), \quad \text{where } X = \mathcal{F}\{x\}, \quad \omega_0 = \frac{2\pi}{N}$$



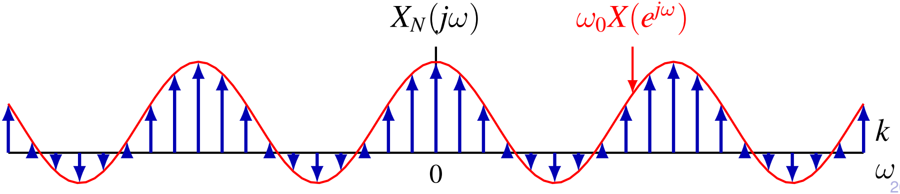
DTFT for Periodic Signals





DTFT $(2\pi)^{-1}X(e^{j\omega})$ is density at frequency ω

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} \hat{x}_N[k]\delta(\omega - k\omega_0) = \omega_0 \sum_{k=-\infty}^{\infty} X(e^{jk\omega_0})\delta(\omega - k\omega_0)$$



DTFT for Periodic Signals

Recall DTFS of x_N

$$x_N[n] = \sum_{k=0}^{N-1} \hat{x}_N[k] e^{j\frac{2k\pi}{N}n}$$

Take DTFT of both sides, use linearity and DTFT of complex exponentials,

$$X_N(e^{j\omega}) = \sum_{k=0}^{N-1} \hat{x}_N[k] \mathcal{F}\{e^{j\frac{2k\pi}{N}n}\} = \sum_{k=0}^{N-1} \hat{x}_N[k] \sum_{\ell=-\infty}^{\infty} 2\pi\delta(\omega - \frac{2k\pi}{N} - 2\ell\pi)$$

$$= 2\pi \sum_{\ell=-\infty}^{\infty} \sum_{k=0}^{N-1} \hat{x}_N[k]\delta(\omega - \frac{2(k+\ell N)\pi}{N})$$

$$= 2\pi \sum_{k=-\infty}^{\infty} \hat{x}_N[k]\delta(\omega - \frac{2k\pi}{N}) \quad \text{(Recall } \hat{x}_N[k] = \hat{x}_N[k+\ell N]\text{)}$$

DTFT for Periodic Signals

Can also verify formally using synthesis equation

$$\frac{1}{2\pi}\int_{-\frac{\pi}{N}}^{2\pi-\frac{\pi}{N}} X(e^{j\omega})e^{j\omega n}d\omega = \int_{-\frac{\pi}{N}}^{2\pi-\frac{\pi}{N}} \sum_{k=-\infty}^{\infty} \hat{x}_N[k]\delta(\omega - \frac{2k\pi}{N})e^{j\omega n}d\omega$$

Only terms with $k = 0, 1, \ldots, N-1$ lies in interval of integration

$$\int_{-\frac{\pi}{N}}^{2\pi-\frac{\pi}{N}} \sum_{k=0}^{N-1} \hat{x}_N[k]\delta(\omega - \frac{2k\pi}{N})e^{j\omega n}d\omega = \sum_{k=0}^{N-1} \hat{x}_N[k]e^{j\frac{2k\pi}{N}n} = x_N[n]$$

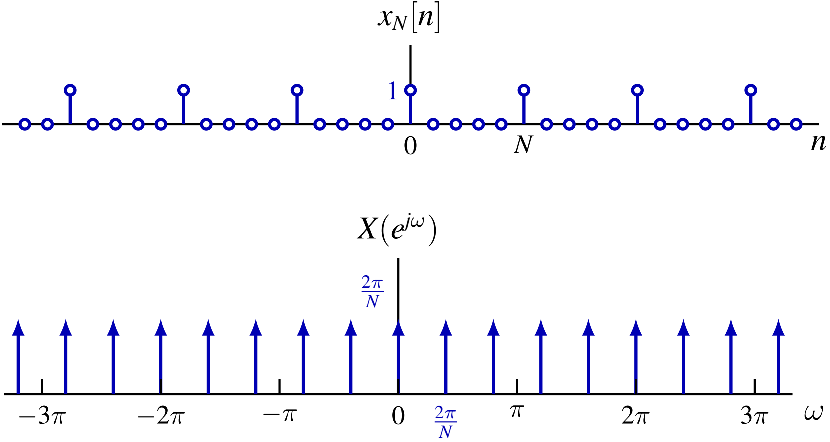
where last equality is DTFS synthesis equation

Thus verified

$$x_N[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Example: DT Periodic Impulse Train

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{N})$$



Contents

Properties of DT Fourier Transform

Periodicity

$$X(e^{j(\omega+2\pi)}) = X(e^{j(\omega)})$$

Linearity

$$\mathcal{F}\{ax + by\} = a\mathcal{F}\{x\} + b\mathcal{F}\{y\}$$

Time and frequency shifting

If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

then

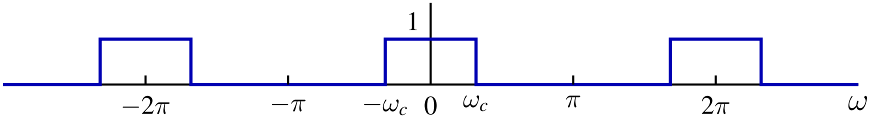
$$x[n - n_0] \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$$

and

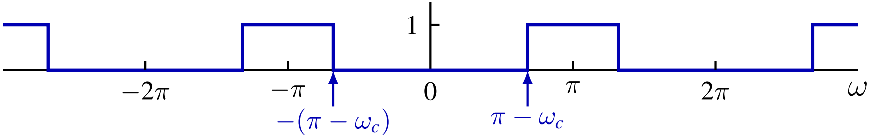
$$e^{j\omega_0 n} x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})$$

Example: Highpass vs. Lowpass Filters

$$H_{\text{lp}}(e^{j\omega})$$



$$H_{\text{hp}}(e^{j\omega}) = H_{\text{lp}}(e^{j(\omega-\pi)})$$



$$H_{\text{hp}}(e^{j\omega}) = H_{\text{lp}}(e^{j(\omega-\pi)}) \iff h_{\text{hp}}[n] = e^{j\pi n} h_{\text{lp}}[n] = (-1)^n h_{\text{lp}}[n]$$

Highpass filtering $y = x * h_{\text{hp}}$ implemented by lowpass filter

(1). $x_1[n] = (-1)^n x[n]$; (2). $y_1 = x_1 * h_{\text{lp}}$; (3). $y[n] = (-1)^n y_1[n]$

Properties of DT Fourier Tranform

Assume

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

Time reversal

$$x[-n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{-j\omega})$$

Conjugation

$$x^*[n] \overset{\mathcal{F}}{\longleftrightarrow} X^*(e^{-j\omega})$$

Symmetry

- x even $\iff$ X even, x odd $\iff$ X odd
- x real $\iff$ $X(e^{-j\omega}) = X^*(e^{j\omega})$
- x real and even $\iff$ X real and even
- x real and odd $\iff$ X purely imaginary and odd

Differencing and Accumulation

First (backward) difference

$$x[n] - x[n-1] \overset{\mathcal{F}}{\longleftrightarrow} (1 - e^{-j\omega})X(e^{j\omega})$$

Accumulation (Running sum)

$$y[n] = \sum_{m=-\infty}^{n} x[m] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

- first term from differencing property
- second term is DTFT of DC component $\mathcal{F}\{\bar{y}\}$, $\bar{y} = \frac{1}{2}X(e^{j0})$; cf. lecture 15, slide 17

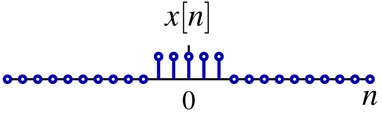
Example. Since $\delta[n] \overset{\mathcal{F}}{\longleftrightarrow} 1$,

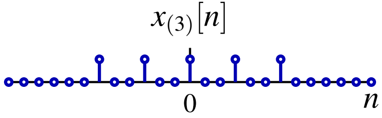
$$u[n] = \sum_{m=-\infty}^{n} \delta[m] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

Time Expansion

Define $x_{(m)}$ by

$$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is multiple of } m \\ 0, & \text{otherwise} \end{cases}$$





If

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

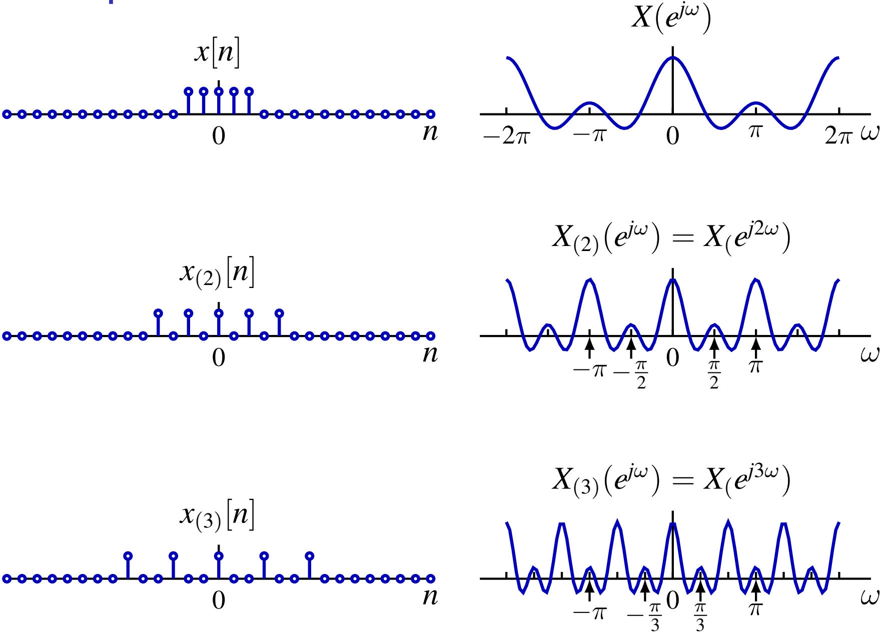
then

$$x_{(m)}[n] \xleftrightarrow{\mathcal{F}} X(e^{jm\omega})$$

Proof.

$$\mathcal{F}\{x_{(m)}\}(e^{jm\omega}) = \sum_{n=-\infty}^{\infty} x_{(m)}[n]e^{-j\omega n} = \sum_{\ell=-\infty}^{\infty} x[\ell]e^{-j\omega m\ell} = X(e^{jm\omega})$$

Example


$x[n]$
0
n
$X(e^{j\omega})$
-2π
$-\pi$
0
π
2π
ω
$x_{(2)}[n]$
0
n
$X_{(2)}(e^{j\omega}) = X_{(}e^{j2\omega})$
$-\pi$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
π
ω
$x_{(3)}[n]$
0
n
$X_{(3)}(e^{j\omega}) = X_{(}e^{j3\omega})$
$-\pi$
$-\frac{\pi}{3}$
0
$\frac{\pi}{3}$
π
ω

Frequency Differentiation

If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

then

$$nx[n] \stackrel{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}X(e^{j\omega})$$

Proof.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Differentiate under summation sign

$$\frac{d}{d\omega}X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](-jn)e^{-j\omega n}$$

Parseval's Identity

Theorem. If $x \in \ell_2$, $X = \mathcal{F}\{x\}$, then

$$\|x\|_{\ell_2}^2 = \|X\|_{L_2(2\pi)}^2, \qquad \text{or} \qquad \sum_{n\in\mathbb{Z}} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Note ω is angular frequency and $\frac{\omega}{2\pi}$ is frequency

$$\sum_{n\in\mathbb{Z}} |x[n]|^2 = \int_{2\pi} |X(e^{j\omega})|^2 \frac{d\omega}{2\pi}$$

Interpretation: Energy conservation

- $\sum_{n\in\mathbb{Z}} |x[n]|^2$ total energy
- $|X(e^{j\omega})|^2$ energy per unit frequency

$|X(e^{j\omega}))|^2$ called energy-density spectrum

Parseval's Identity

Theorem. If $x, y \in \ell_2$, $X = \mathcal{F}\{x\}$, $Y = \mathcal{F}\{y\}$, then

$$\langle x, y\rangle_{\ell_2} = \langle X, Y\rangle_{L_2(2\pi)}, \qquad \text{or} \qquad \sum_{n\in\mathbb{Z}} x[n]y^*[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

Proof.

$$\begin{aligned}
\langle X, Y\rangle_{L_2(2\pi)} &= \langle \sum_{n\in\mathbb{Z}} x[n]e^{-j\omega n}, \sum_{m\in\mathbb{Z}} y[m]e^{-j\omega m}\rangle_{L_2(2\pi)} \\
&= \sum_{n\in\mathbb{Z}} x[n] \sum_{m\in\mathbb{Z}} y^*[m]\langle e^{-j\omega n}, e^{-j\omega m}\rangle_{L_2(2\pi)} \\
&= \sum_{n\in\mathbb{Z}} x[n] \sum_{m\in\mathbb{Z}} y^*[m]\delta[n-m] \\
&= \sum_{n\in\mathbb{Z}} x[n]y^*[n] = \langle x, y\rangle_{\ell_2}
\end{aligned}$$