EI331 Signals and Systems

Lecture 20

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1. Magnitude-phase Representation of Fourier Transform

2. Uncertainty Principle

3. Relations Among Fourier Representations

Magnitude-phase Representation of Fourier Transform

$$X(j\omega) = |X(j\omega)| e^{j \arg X(j\omega)}, \quad X(e^{j\omega}) = |X(e^{j\omega})| e^{j \arg X(e^{j\omega})}$$

Recall Fourier transform is decomposition of signal into superposition of complex exponentials (“waves”)

- $|X|$ gives magnitudes of components
- $\arg X$ gives phases of components

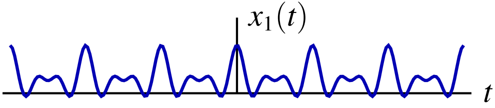
Phase $\arg X$ contains **substantial** information about signal

- determines whether components add constructively or destructively
- small change can lead to very differential-looking signals for same magnitude spectrum

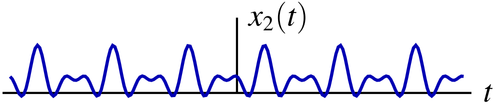
Importance of Phase Information

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t + \phi_1) + \cos(4\pi t + \phi_2) + \frac{2}{3}\cos(6\pi t + \phi_3)$$

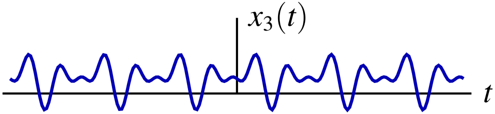
$\phi_1 = 0, \phi_2 = 0, \phi_3 = 0$



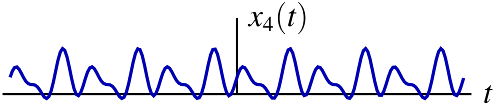
$\phi_1 = 4, \phi_2 = 8, \phi_3 = 12$



$\phi_1 = 6, \phi_2 = -2.7, \phi_3 = 0.93$

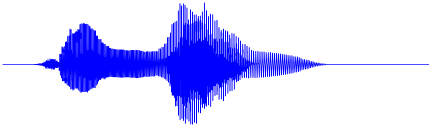


$\phi_1 = 1.2, \phi_2 = 4.1, \phi_3 = -7.02$

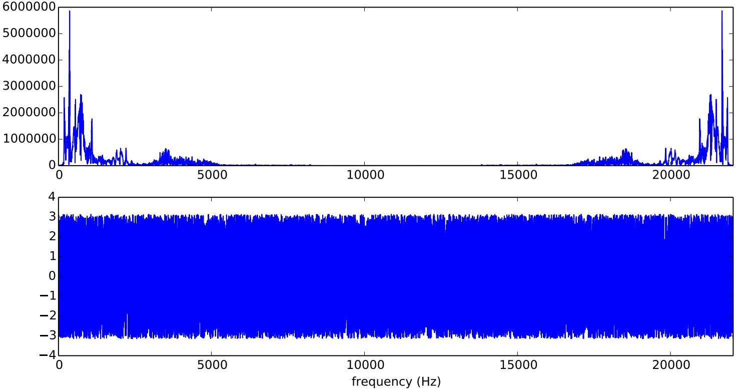


Magnitude vs. Phase

Waveform x for Chinese word “中文”

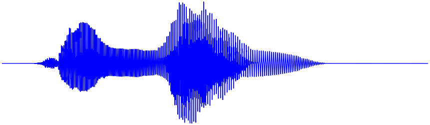


Magnitude and phase spectra $|X|$, $\arg X$ (DFT)

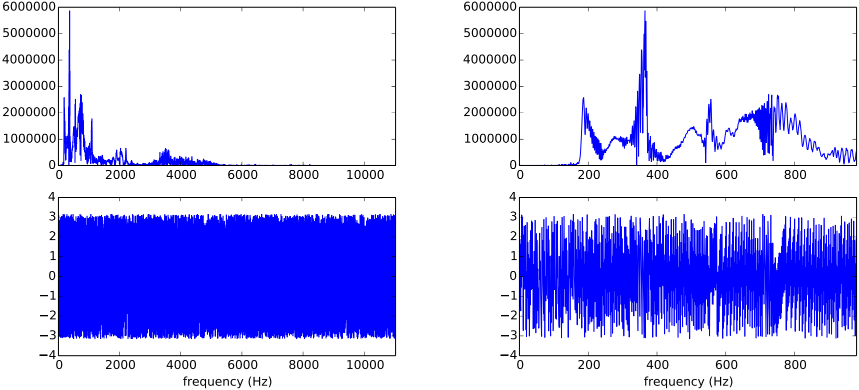


Magnitude vs. Phase

Waveform x for Chinese word “中文”

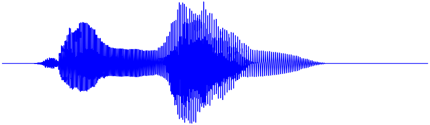


Magnitude and phase spectra $|X|$, $\arg X$ (DFT)

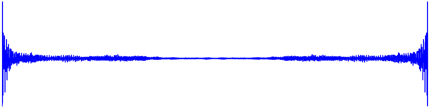


Magnitude vs. Phase

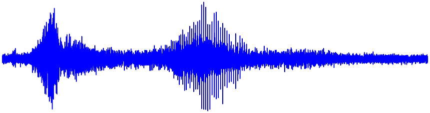
Waveform x for Chinese word “中文”



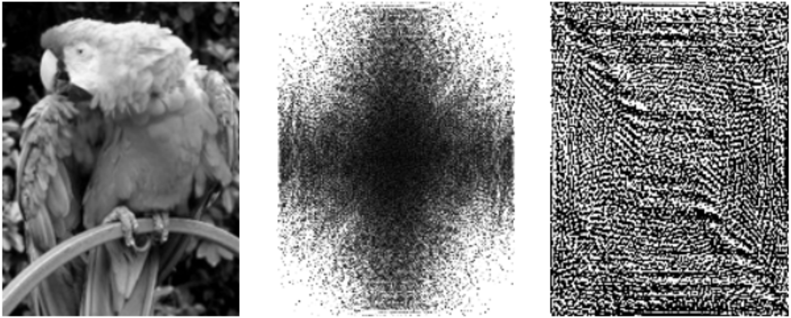
Waveform reconstructed by magnitude spectra only $\mathcal{F}^{-1}\{|X|\}$

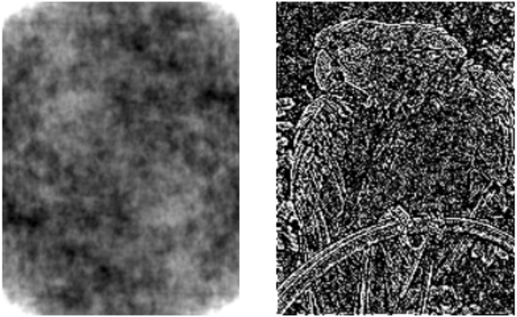


Waveform reconstructed by phase spectra only $\mathcal{F}^{-1}\{e^{j \arg X}\}$



Magnitude vs. Phase





Top row

X, $|X|$, $\arg X$

Bottom row

$\mathcal{F}^{-1}\{|X|\}$

$\mathcal{F}^{-1}\{e^{j \arg X}\}$

Magnitude-phase Representation of Frequency Response

For LTI systems

$$Y(j\omega) = H(j\omega)X(j\omega), \quad Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

Thus

$|Y| = |H| \cdot |X|,$ $|H|$ called gain of system

and

$\arg Y = \arg H + \arg X,$ $\arg H$ called phase shift of system

Effects of LTI system may or may not be desirable

- want specific effects for filtering
- if undesirable, effects called distortion

Example. Distortionless transmission

- ideally, $H(j\omega) = 1$, but noncausal
- $H(j\omega) = Ke^{-j\omega t_0}$, preserves shape, only scaling + delay

Linear Phase

For CT LTI system with unit gain and linear phase

$$H(j\omega) = e^{-j\omega t_0} \implies y(t) = x(t - t_0)$$

output is delayed version of input

For DT LTI system with unit gain and linear phase

$$H(e^{j\omega}) = e^{-j\omega n_0},$$

output

$$y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{-j\omega n_0} e^{j\omega n} d\omega = \sum_{m=-\infty}^{\infty} x[m] \operatorname{sinc}(n - n_0 - m)$$

- for integer n_0, $y[n] = x[n - n_0]$ is delayed version of input
- for non-integer n_0, $y[n] = y_c(n - n_0)$ is sample of delayed version of envelope $y_c(t) = \sum_{m=-\infty}^{\infty} x[m] \operatorname{sinc}(t - m)$ of x

Linear Phase

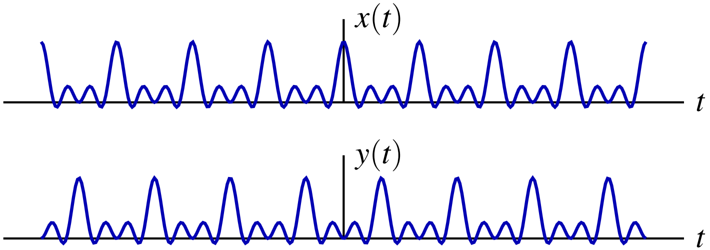
$$H(j\omega) = e^{-j\omega/2}$$

For input

$$x(t) = \sum_{k=0}^{3} \cos(2k\pi t) = 1 + \cos(2\pi t) + \cos(4\pi t) + \cos(6\pi t)$$

output is

$$y(t) = \sum_{k=0}^{3} \cos(2k\pi t - k\pi) = x(t - \frac{1}{2})$$



Linear Phase

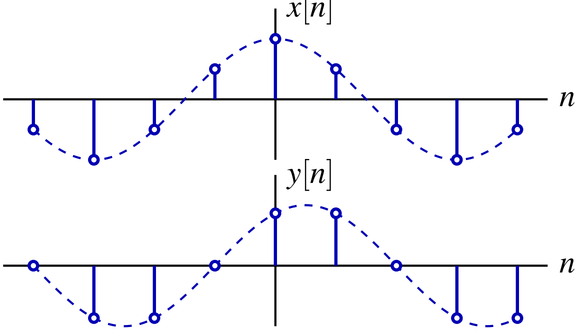
Half-sample delay $$H(e^{j\omega}) = e^{-j\omega/2}$$

For input

$$x[n] = \cos(\frac{\pi}{3}n)$$

output is

$$y[n] = \sum_{m=-\infty}^{\infty} \cos(\frac{\pi}{3}m)\,\mathrm{sinc}(n - n_0 - m) = \cos(\frac{\pi}{3}[n - \frac{1}{2})]$$



Nonlinear Phase

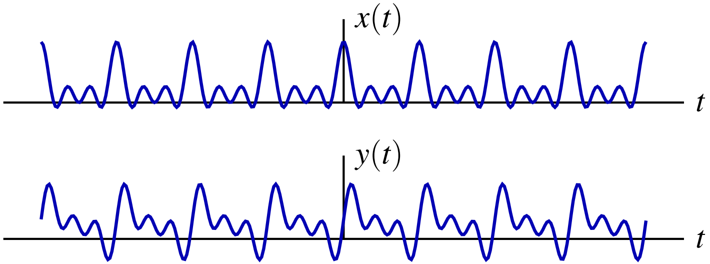
$$H(j\omega) = e^{-j \arctan \omega}$$

For input

$$x(t) = \sum_{k=0}^{3} \cos(2k\pi t) = 1 + \cos(2\pi t) + \cos(4\pi t) + \cos(6\pi t)$$

output is

$$y(t) = \sum_{k=0}^{3} \cos[2k\pi t - \arctan(2k\pi)]$$



Group Delay

For narrowband input x centered at ω_0, i.e.

$$X(j\omega) = 0, \quad \text{for } |\omega - \omega_0| > \Delta\omega, \text{ where } \Delta\omega \ll 1$$

Use linear approximation for phase

$$\arg H(j\omega) \approx \arg H(j\omega_0) - \tau(\omega_0)(\omega - \omega_0) = \phi_0 - \tau(\omega_0)\omega$$

where group delay at ω is

$$\tau(\omega) = -\frac{d}{d\omega} \arg H(j\omega)$$

If $|H(j\omega)| \approx |H(j\omega_0)|$ for $|\omega - \omega_0| \leq \Delta\omega$,

$$Y(j\omega) \approx |H(j\omega_0)| X(j\omega) e^{j\phi_0 - j\tau(\omega_0)\omega} \implies y(t) \approx |H(j\omega_0)| e^{j\phi_0} x(t - \tau(\omega_0))$$

Group Delay

$$H(j\omega) = e^{-j\arctan\omega} \implies \tau(\omega) = \frac{d}{d\omega}\arctan(\omega) = \frac{1}{1+\omega^2}$$

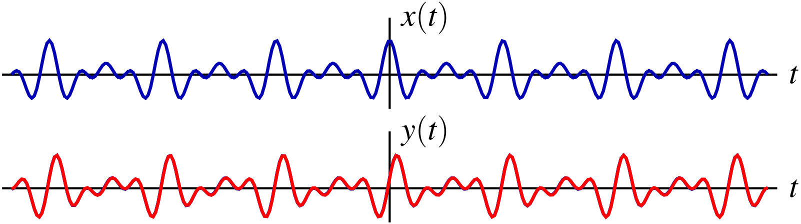
For input (sum of **two** narrowband signals z, $\bar{z}$ centered at $\pm\pi$)

$$x(t) = \sum_{k=9}^{11}\cos(\frac{k\pi}{10}t) = \text{Re}\, z(t) = \frac{1}{2}z(t)+\frac{1}{2}\bar{z}(t), \text{where } z(t) = \sum_{k=9}^{11} e^{j\frac{k\pi}{10}t}$$

output

$$y(t) \approx \frac{1}{2}e^{j\phi_0}z(t-\tau_0) + \frac{1}{2}e^{-j\phi_0}\bar{z}(t-\tau_0) = \text{Re}\left[e^{j\phi_0}z(t-\tau_0)\right]$$

where $\phi_0 = -\arctan\pi + \frac{\pi}{1+\pi^2}$, $\tau_0 = \frac{1}{1+\pi^2}$.



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3. Relations Among Fourier Representations

Uncertainty Principle

Assume CT signal $x \in L_2(\mathbb{R})$, so $X = \mathcal{F}\{x\} \in L_2(\mathbb{R})$

Define normalized power density in time and frequency

$$p(t) = \frac{|x(t)|^2}{\int_{\mathbb{R}} |x(\tau)|^2 d\tau}, \quad P(\omega) = \frac{|X(j\omega)|^2}{\int_{\mathbb{R}} |X(j\theta)|^2 d\theta}$$

NB. p and P can be interpreted as probability densities, as done in quantum mechanics

x and X are centered at t_0 and ω_0 resp. in the sense

$$t_0 = \int_{\mathbb{R}} tp(t)dt, \quad \omega_0 = \int_{\mathbb{R}} \omega P(\omega)d\omega$$

"Standard deviation" measures energy spread around center

$$\Delta t = \left[\int_{\mathbb{R}} (t - t_0)^2 p(t)dt\right]^{\frac{1}{2}}, \quad \Delta\omega = \left[\int_{\mathbb{R}} (\omega - \omega_0)^2 P(\omega)d\omega\right]^{\frac{1}{2}}$$

Uncertainty Principle

Theorem. If $x(t) \in L_2(\mathbb{R})$ with Fourier transform $X(j\omega)$, then

$$\Delta t \Delta\omega \geq \frac{1}{2}$$

with equality iff x is Gaussian

In fact, the following slightly more general relation holds

$$D_a(x) D_b(X) \geq \frac{1}{2} \|x\|_2 \cdot \|X\|_2$$

where for $g \in L_2(\mathbb{R})$ and $a \in \mathbb{R}$,

$$D_a(g) = \left[\int_{\mathbb{R}} (\xi - a)^2 |g(\xi)|^2 d\xi \right]^{\frac{1}{2}}$$

NB. Roughly speaking, signals cannot be localized in both time and frequency; short pulse has large bandwidth, narrowband signal has long duration

Proof of Uncertainty Principle

First assume $a = b = 0$.

$$D_0(X) = \left[\int_{\mathbb{R}} \omega^2 |X(j\omega)|^2 dt\right]^{\frac{1}{2}} = \left[\int_{\mathbb{R}} |j\omega X(j\omega)|^2 d\omega\right]^{\frac{1}{2}}$$

Since $x'(t) \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega)$, Parseval's identity yields

$$D_0(X) = \left[2\pi \int_{\mathbb{R}} |x'(t)|^2 dt\right]^{\frac{1}{2}}$$

By Cauchy-Schwarz inequality

$$\begin{aligned} D_0(x)D_0(X) &= \sqrt{2\pi}\left[\int_{\mathbb{R}} |tx(t)|^2 dt\right]^{\frac{1}{2}} \left[\int_{\mathbb{R}} |x'(t)|^2 dt\right]^{\frac{1}{2}} \\ &\geq \sqrt{2\pi}\left|\int_{\mathbb{R}} tx^*(t)x'(t)dt\right| \geq \sqrt{2\pi}\left|\text{Re} \int_{\mathbb{R}} tx^*(t)x'(t)dt\right| \end{aligned}$$

Proof of Uncertainty Principle

Note

$$2\text{Re}\int_{\mathbb{R}} tx^*(t)x'(t)dt = \int_{\mathbb{R}} tx^*(t)x'(t)dt + \int_{\mathbb{R}} tx(t)\overline{x'(t)}dt = \int_{\mathbb{R}} td|x(t)|^2$$

Integration by parts yields

$$2\text{Re}\int_{\mathbb{R}} tx^*(t)x'(t)dt = t|x(t)|^2\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} |x(t)|^2 dt$$

Since inequality is trivial if $D_0(x) = \infty$, can assume $D_0(x) < \infty$, so $t|x(t)|^2 \to 0$ as $t \to \infty$. Thus

$$D_0(x)D_0(X) \geq \frac{\sqrt{2\pi}}{2}\|x\|_2^2 = \frac{1}{2}\|x\|_2 \cdot \|X\|_2$$

For $a, b \neq 0$, note $D_a(x) = D_0(y)$ and $D_b(X) = D_0(Y)$ for
$y(t) = x(t+a)e^{-jbt} \stackrel{\mathcal{F}}{\longleftrightarrow} Y(j\omega) = X(j(\omega+b))e^{j(\omega+b)a}$

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Four Fourier Representations

CT Fourier series

$$\hat{x}[k] = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi}{T}kt}dt$$

$$x(t) = x(t+T) = \sum_{k\in\mathbb{Z}} \hat{x}[k]e^{j\frac{2\pi}{T}kt}$$

CT Fourier transform

$$X(j\omega) = \int_{\mathbb{R}} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{\mathbb{R}} X(j\omega)e^{j\omega t}d\omega$$

DT Fourier series

$$\hat{x}[k] = \frac{1}{N}\sum_{n\in[N]} x[n]e^{-j\frac{2\pi}{N}kn}$$

$$x[n] = x[n+N] = \sum_{k\in[N]} \hat{x}[k]e^{j\frac{2\pi}{N}kn}$$

DT Fourier transform

$$X(e^{j\omega}) = \sum_{n\in\mathbb{Z}} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

DFT is one period of DTFS

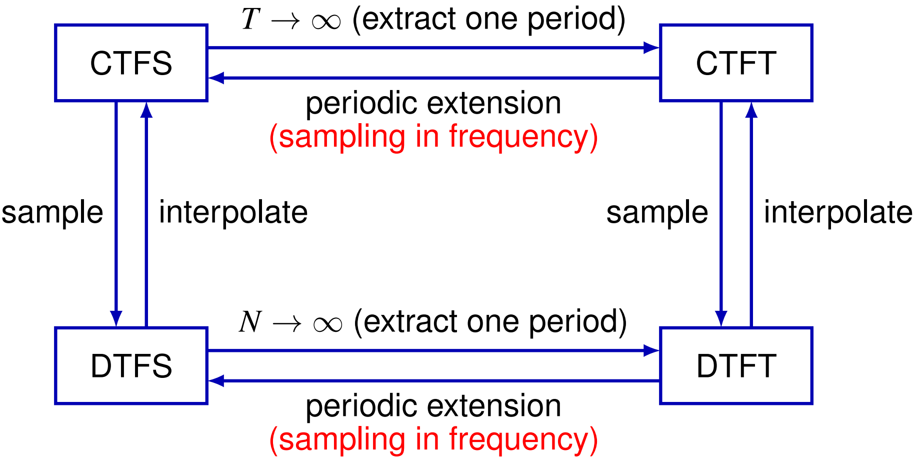
Relations among Four Fourier Representations

	time		frequency	
CTFS	continuous	periodic	discrete	aperiodic
CTFT	continuous	aperiodic	continuous	aperiodic
DTFS	discrete	periodic	discrete	periodic
DTFT	discrete	aperiodic	continuous	periodic

Observations

- periodic in one domain $\iff$ discrete in other domain
- discretization by sampling in one domain $\iff$ periodic extension in other domain
- continualization by interpolation in one domain $\iff$ extraction of one period in other domain

Relations among Four Fourier Representations



NB. Conditions apply in some cases.