EI331 Signals and Systems

Lecture 22

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Contents

Derivative and Differential of Complex Functions

Suppose $w = f(z)$ is defined on a domain $D \subset \mathbb{C}$ and $z_0 \in D$. If the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists, then we call it the derivative of f at z_0, and write

$$f'(z_0) = \frac{df}{dz}\Big|_{z=z_0} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

If the increment of $f(z)$ at z_0 can be written as

$$\Delta f(z_0) = f(z_0 + \Delta z) - f(z_0) = A\Delta z + \alpha(z)\Delta z$$

where $A \in \mathbb{C}$ is a constant, and $\alpha(z) \to 0$ as $\Delta z \to 0$, we say f is differentiable at z_0, and call $A\Delta z$ the differential of f at z_0.

Analytic Functions

If f is differentiable for every z in an open disk $B(z_0, \delta)$, then we say f is analytic at z_0.

If f is differentiable for every z in a domain D, then we say f is an analytic (or holomorphic) function on D.

Example. $f(z) = z^2$ analytic on $\mathbb{C}$.

Example. $f(z) = \frac{1}{z}$ is differentiable at every $z \neq 0$ with derivative $f'(z) = -\frac{1}{z^2}$, so f is analytic on $\mathbb{C} \setminus \{0\}$.

For a complex function f, the following entailments hold

$$\text{analytic at } t_0 \implies \text{differentiable at } t_0 \implies \text{continuous at } t_0$$

Example. $f(z) = \bar{z}$ is continuous on $\mathbb{C}$ but nowhere differentiable.

Example. $f(z) = (\text{Re}\, z)^2$ is differentiable but not analytic at points on the imaginary axis.

Analytic Functions

Example. $f(z) = (\text{Re}\, z)^2$ is differentiable but not analytic on the imaginary axis.

Proof. Let $z_0 = x_0 + jy_0$, $\Delta z = \Delta x + j\Delta y$ and $z = z_0 + \Delta z$.

1. If $x_0 = 0$, since $\Delta x \leq \Delta z$
$$\left|\frac{f(z) - f(z_0)}{z - z_0} - 0\right| = \left|\frac{(\Delta x)^2}{\Delta z}\right| \leq |\Delta z| \implies f'(z_0) = 0$$

2. If $x_0 \neq 0$, let $\Delta z \to 0$ along the real and imaginary axes,
$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{2x_0\Delta x + (\Delta x)^2}{\Delta x + j\Delta y} \xrightarrow{\Delta z \to 0} \begin{cases} 2x_0 \neq 0, & \text{along } \Delta y = 0 \\ 0, & \text{along } \Delta x = 0 \end{cases}$$
So $f'(z_0)$ does not exist if $x_0 \neq 0$.

3. Since any open disk $B(z_0, \delta)$ contains points z with $\text{Re}\, z \neq 0$, f is not analytic at any point $z_0 \in \mathbb{C}$.

Analytic Functions

The rules for taking derivatives imply the following theorems.

Theorem. If f and g are analytic at z_0, then so are $f \pm g$, fg and f/g (if $g(z_0) \neq 0$).

NB. By definition, f and g are differentiable on some $B(z_0, \delta_1)$ and $B(z_0, \delta_2)$, respectively. For $f \pm g$, fg and f/g, we can always take $B(z_0, \delta)$, where $\delta = \min\{\delta_1, \delta_2\}$.

Theorem. If $h = g(z)$ is analytic at z_0, $w = f(h)$ is analytic at $h_0 = g(z_0)$, then $w = f \circ g(z)$ is analytic at z_0.

Example. A polynomial $P(z) = \sum_{k=0}^{n} a_k z^k$ is analytic on $\mathbb{C}$.

Example. A rational function $R(z) = \dfrac{P(z)}{Q(z)}$ is analytic on $\mathbb{C} \setminus \{z : Q(z) = 0\}$, where P, Q are polynomials.

Cauchy-Riemann Equations

Recall the continuity of $f(z) = u(x,y) + jv(x,y)$ at $z_0 = x_0 + jy_0$ is equivalent to the continuity of $u(x,y)$ and $v(x,y)$ at (x_0, y_0).

The differentiability of $f(z) = u(x,y) + jv(x,y)$ at $z_0 = x_0 + jy_0$ is **not equivalent** to the differentiability of $u(x,y)$ and $v(x,y)$ at (x_0, y_0).

Example. $u(x,y) = x^2$ and $v(x,y) = 0$ are differentiable on the entire $\mathbb{R}^2$, but $f(z) = (\mathrm{Re}\, z)^2 = u(x,y) + jv(x,y)$ is differentiable only on the imaginary axis.

Cauchy-Riemann equations

Let $\Delta z \to 0$ along the real and imaginary axes, i.e. $\Delta z = \Delta x$ and $\Delta z = j\Delta y$, respectively,

$$f'(z) = \frac{\partial u(x,y)}{\partial x} + j\frac{\partial v(x,y)}{\partial x} = -j\frac{\partial u(x,y)}{\partial y} + \frac{\partial v(x,y)}{\partial y}$$

$$\Longrightarrow \quad \frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y}, \qquad \frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x}$$

Necessary & Sufficient Conditions for Differentiability

Theorem. A function $f(z) = u(x,y) + jv(x,y)$ defined on a domain D is differentiable at $z_0 = x_0 + jy_0 \in D$ if and only if

1. $u(x,y)$ and $v(x,y)$ are (real) differentiable at (x_0, y_0)
2. the partial derivatives satisfy the Cauchy-Riemann equations at (x_0, y_0),

$$\frac{\partial u(x_0,y_0)}{\partial x} = \frac{\partial v(x_0,y_0)}{\partial y}, \quad \frac{\partial u(x_0,y_0)}{\partial y} = -\frac{\partial v(x_0,y_0)}{\partial x}$$

Proof. For necessity, assume $f'(z_0) = a + jb$ exists. By definition,

$$\Delta f(z_0) = f'(z_0)\Delta z + \alpha(\Delta z), \text{ where } \alpha(\Delta z) = o(\Delta z)$$

Let $\alpha(\Delta z) = \alpha_1(\Delta z) + j\alpha_2(\Delta z)$. Then $\alpha_i(\Delta z) = o(\Delta z)$. Note

$$\Delta u(x_0,y_0) = \text{Re}\,\Delta f(z_0) = (a\Delta x - b\Delta y) + \alpha_1(\Delta z)$$
$$\Delta v(x_0,y_0) = \text{Im}\,\Delta f(z_0) = (a\Delta y + b\Delta x) + \alpha_2(\Delta z)$$

so u, v are differentiable and $a = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $-b = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Necessary & Sufficient Conditions for Differentiability

Proof (cont'd). For sufficiency, assume u, v are differentiable, and the Cauchy-Riemann equations hold. So

$$\Delta u(x_0, y_0) = (a\Delta x - b\Delta y) + \alpha_1(\Delta z)$$
$$\Delta v(x_0, y_0) = (a\Delta y + b\Delta x) + \alpha_2(\Delta z),$$

where $a = \frac{\partial u(x_0,y_0)}{\partial x} = \frac{\partial v(x_0,y_0)}{\partial y}$, $-b = \frac{\partial u(x_0,y_0)}{\partial y} = -\frac{\partial v(x_0,y_0)}{\partial x}$, and $\alpha_i(\Delta z) = o(|\Delta z|)$, $i = 1, 2$.

Thus

$$\Delta f = \Delta u + j\Delta v = (a + jb)\Delta z + \alpha(\Delta z)$$

where $\alpha(\Delta z) = \alpha_1(\Delta z) + j\alpha_2(\Delta z)$. Note $\alpha(\Delta z) = o(\Delta z)$, since

$$\left|\frac{\alpha(\Delta z)}{\Delta z}\right| \leq \frac{|\alpha_1(\Delta z)|}{|\Delta z|} + \frac{|\alpha_2(\Delta z)|}{|\Delta z|} \to 0, \text{ as } \Delta z \to 0.$$

So f is differentiable with

$$f'(z_0) = u_x(x_0, y_0) - ju_y(x_0, y_0) = v_y(x_0, y_0) + jv_x(x_0, y_0)$$

Example

The Jacobian matrix of f viewed as a mapping $(x, y) \mapsto (u, v)$ is

$$J[f] = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

Mnemonics for the sign in the Cauchy-Riemann equations:

- Entries on the principal diagonal are identical, $u_x = v_y$
- Entries on the secondary diagonal differ in signs, $u_y = -v_x$

Exmaple. $f(z) = \bar{z} = x - jy$ with $u(x, y) = x$ and $v(x, y) = -y$.

$$J[f] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Since $u_x \neq v_y$, one of the Cauchy-Riemann equations fails everywhere, so f is nowhere differentiable.

Examples

Exmaple. $f(z) = e^z = e^x(\cos y + j\sin y)$ with $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$.

$$J[f] = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

The partial derivatives are all continuous, so u and v are differentiable on $\mathbb{R}^2$. Since the Cauchy-Riemann equations also hold, f is differentiable and hence analytic on $\mathbb{C}$ with $f'(z) = f(z)$.

Exmaple. $f(z) = z\text{Re}\, z$ with $u(x,y) = x^2$ and $v(x,y) = xy$.

$$J[f] = \begin{pmatrix} 2x & 0 \\ y & x \end{pmatrix}$$

u and v are differentiable on $\mathbb{R}^2$. Since the Cauchy-Riemann equations hold only if $x = y = 0$, f is differentiable only at $z = 0$ and nowhere analytic on $\mathbb{C}$.

Necessary & Sufficient Conditions for Analyticity

Theorem. A function $f(z) = u(x,y) + jv(x,y)$ is analytic on a domain D if and only if

1. $u(x,y)$ and $v(x,y)$ are (real) differentiable on D
2. the Cauchy-Riemann equations hold on D

We will see later analytic functions are infinitely differentiable. Since f'' exists, all partial derivatives are continuous.

Theorem. A function $f(z) = u(x,y) + jv(x,y)$ is analytic on a domain D if and only if

1. $u(x,y)$ and $v(x,y)$ are continuously differentiable on D
2. the Cauchy-Riemann equations hold on D

Corollary. If $f(z) = u(x,y) + jv(x,y)$ is analytic on D, then u and v are harmonic functions on D, i.e. $u_{xx} + u_{yy} = 0$, $v_{xx} + v_{yy} = 0$.

Corollary. If f has vanishing derivative on a domain D, i.e. $f'(z) = 0$ on D, then f is a constant on D.

Analytic Function as "True" Function of z

Recall

$$\begin{cases} x = \frac{1}{2}(z + \bar{z}) \\ y = \frac{1}{2j}(z - \bar{z}) \end{cases}$$

If we view z and $\bar{z}$ as two independent variables, then by the chain rule,

$$\frac{\partial f}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2j}\frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2j}\frac{\partial f}{\partial y}$$

Since $f = u + jv$,

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{j}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

The Cauchy-Riemann equations are equivalent to $\frac{\partial f}{\partial \bar{z}} = 0$. Thus an analytic function depends only on z and **not** on $\bar{z}$.

Analytic Function as "True" Function of z

Theorem. A function $f(z) = u(x, y) + jv(x, y)$ is analytic on a domain D if and only if

1. $u(x, y)$ and $v(x, y)$ are (real) differentiable on D
2. $\dfrac{\partial f}{\partial \bar{z}} = 0$ on D

Example. $f(z) = (\text{Re}\, z)^2$.

$$f(z) = \left(\frac{z + \bar{z}}{2}\right)^2 \implies \frac{\partial f}{\partial \bar{z}} = \frac{z + \bar{z}}{2} \neq 0 \text{ if } \text{Re}\, z \neq 0$$

So f is nowhere analytic.

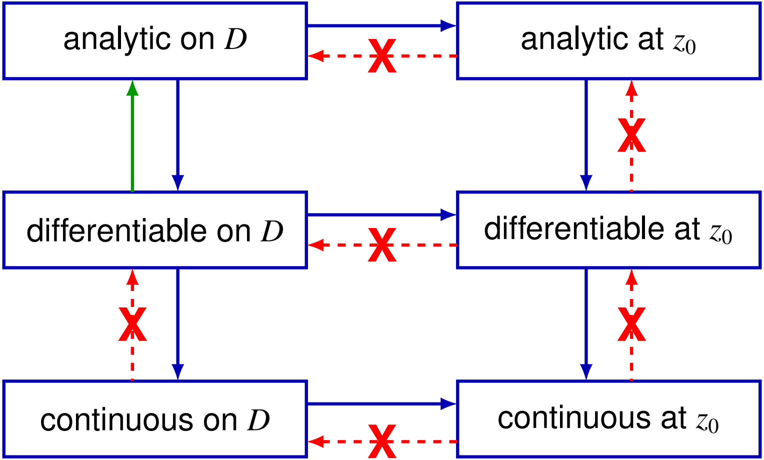
Example. $f(z) = |z|^2$.

$$f(z) = z\bar{z} \implies \frac{\partial f}{\partial \bar{z}} = z \neq 0 \text{ if } z \neq 0$$

So f is nowhere analytic.

Relations: Continuity, Differentiability and Analyticity

f is defined on a domain D and $z_0 \in D$.



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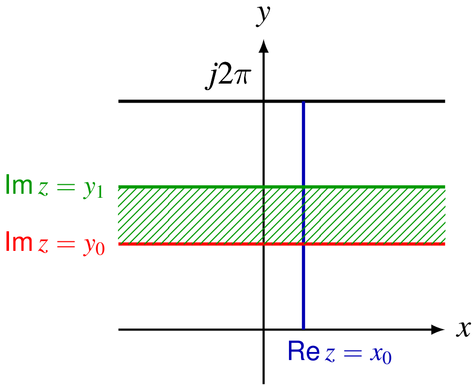
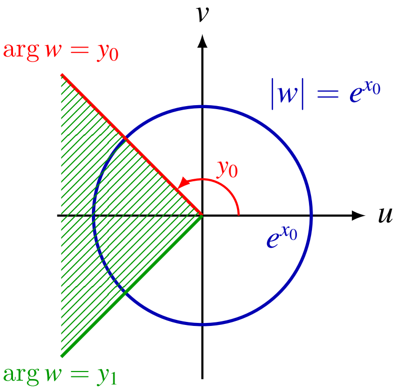
Exponential

$$e^z = \exp z = e^x(\cos y + j\sin y), \quad (e^z \text{ is } \textbf{not } e \text{ to the power of } z)$$

- $e^z \neq 0$
- $e^{z_1+z_2} = e^{z_1}e^{z_2}$
- $(e^z)^{-1} = e^{-z}$
- $\overline{e^z} = e^{\bar{z}}$
- periodic $e^{z+j2\pi} = e^z$
- e^z is analytic on $\mathbb{C}$ and $(e^z)' = e^z$





Trigonometric Functions

$$\sin z = \frac{e^{jz} - e^{-jz}}{2j}, \quad \cos z = \frac{e^{jz} + e^{-jz}}{2}$$

Many properties of real $\sin$ and $\cos$ remain true

- $\sin z$ and $\cos z$ are analytic on $\mathbb{C}$

$$(\sin z)' = \cos z, \quad (\cos z)' = -\sin z$$

- $\sin z$ and $\cos z$ are periodic with period 2π

$$\sin(z + 2\pi) = \sin z, \quad \cos(z + 2\pi) = \cos z$$

- $\sin z$ is odd and $\cos z$ is even

$$\sin(-z) = -\sin z, \quad \cos(-z) = \cos z$$

- $$\sin^2 z + \cos^2 z = 1, \quad \sin(\frac{\pi}{2} - z) = \cos z$$

Trigonometric Functions

- $$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$
  $$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

In particular, for $x, y \in \mathbb{R}$,

$$\sin(x + jy) = \sin x \cos(jy) + \cos x \sin(jy)$$

$$\cos(x + jy) = \cos x \cos(jy) - \sin x \sin(jy)$$

By definition,

$$\sin(jy) = \frac{e^{-y} - e^{y}}{2j} = j \sinh y, \quad \cos(jy) = \frac{e^{-y} + e^{y}}{2} = \cosh y$$

so

$$\sin(x + jy) = \sin x \cosh y + j \cos x \sinh y$$

$$\cos(x + jy) = \cos x \cosh y - j \sin x \sinh y$$

Trigonometric Functions

- $\sin z$ and $\cos z$ are unbounded on $\mathbb{C}$. In particular, for $z = jy$, as $y \to \infty$,

$$|\sin(jy)| = |\sinh y| \to \infty, \quad |\cos(jy)| = \cosh y \to \infty$$

- $\sin z = 0$ iff $z = k\pi (k \in \mathbb{Z})$, $\cos z = 0$ iff $z = \frac{\pi}{2} + k\pi (k \in \mathbb{Z})$
  - $\sin(x + jy) = \sin x \cosh y - j \cos x \sinh y = 0$
  - $\sin x \cosh y = 0$ and $\cosh \geq 1 \implies \sin x = 0 \implies x = k\pi$
  - $\cos x \sinh y = 0$ and $x = k\pi \implies \sinh y = 0 \implies y = 0$
- Other trigonometric functions

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}$$

$$\sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}$$

Hyperbolic Functions

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cos z = \frac{e^z + e^{-z}}{2}$$

- $\sinh z$ and $\cosh z$ are analytic on $\mathbb{C}$

$$(\sinh z)' = \cosh z, \quad (\cosh z)' = \sinh z$$

- $\sinh z$ and $\cosh z$ are periodic with period $j2\pi$

$$\sinh(z + j2\pi) = \sinh z, \quad \cosh(z + j2\pi) = \cosh z$$

- Many properties are similar to those of $\sin z$ and $\cos z$

$$\cosh(jz) = \cos z, \quad \cos(jz) = \cosh z$$

$$\sinh(jz) = j \sin z, \quad \sin(jz) = j \sinh z$$

Logarithm

Logarithm $w = \operatorname{Log} z$ is the inverse of exponential.

- By definition, $w = \operatorname{Log} z$ is the root of $e^w = z$
- Since $e^w \neq 0$, $\operatorname{Log} z$ is defined only for $z \neq 0$
- Let $z = re^{j\theta}$, $w = u + jv$. Then $e^{u+jv} = re^{j\theta} \implies r = e^u, e^{jv} = e^{j\theta}$

  Thus
  $$w = \log r + j(\theta + 2k\pi), \quad k \in \mathbb{Z}$$
  or
  $$\operatorname{Log} z = \log |z| + j \operatorname{Arg} z$$

$\operatorname{Log} z$ is a multivalued function. Each $z \neq 0$ has infinitely many logarithms that differ by multiples of $j2\pi$.

When $\operatorname{Arg} z$ is restricted to its principal value $\arg z \in (-\pi, \pi]$, $(\log z)_0 = \log z = \log |z| + j \arg z$ is the principal branch of $\operatorname{Log} z$.

The other branches are $(\log z)_k = \log z + j2k\pi$, $k \in \mathbb{Z}$.

Logarithm

Example. $\operatorname{Log} 2 = \log 2 + j2k\pi$, $k \in \mathbb{Z}$, its principal value is $\log 2$

Example. $\operatorname{Log}(-1) = \log|-1| + j\operatorname{Arg}(-1) = j\pi + j2k\pi$, $k \in \mathbb{Z}$, its principal value is $j\pi$

Example. $\operatorname{Log} j = \log|j| + j\operatorname{Arg} j = j\frac{\pi}{2} + j2k\pi$, $k \in \mathbb{Z}$, its principal value is $j\frac{\pi}{2}$

Example. $\operatorname{Log}(2e^{j\frac{\pi}{4}}) = \log 2 + j\frac{\pi}{4} + j2k\pi$, $k \in \mathbb{Z}$, its principal value is $\log 2 + j\frac{\pi}{4}$

NB.

- Negative real numbers have logarithms.
- $e^{\operatorname{Log} z} = z$, but $\operatorname{Log} e^z = z + j2\pi\mathbb{Z} \neq z$
- $\log e^z \neq z$ in general, e.g. $\log e^{j3\pi} = \log(-1) = j\pi \neq j3\pi$

Logarithm

Some properties of real logarithm are still true

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2$$

$$\operatorname{Log} \frac{z_1}{z_2} = \operatorname{Log} z_1 - \operatorname{Log} z_2$$

- Equality should be interpreted as equality of sets
  - $\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2 = \log |z_1 z_2| + j \arg(z_1 z_2) + j2\pi\mathbb{Z}$
- Equality holds for Log, not for $\log$ in general
  - $\log(-1) = j\pi$, $\log(-1)^2 = 0 \neq \log(-1) + \log(-1) = j2\pi$

The following property of real logarithm is no longer true

$$\operatorname{Log} z^n \neq n \operatorname{Log} z_2$$

- e.g. $\operatorname{Log} j^2 = \operatorname{Log}(-1) = j\pi + j2k\pi \neq 2 \operatorname{Log} j = j\pi + j4k\pi$
- $\operatorname{Log} z^2 = \operatorname{Log} z + \operatorname{Log} z$, but $\operatorname{Log} z + \operatorname{Log} z \neq 2 \operatorname{Log} z$

Logarithm

For $n \in \mathbb{N}$,

$$\operatorname{Log} \sqrt[n]{z} = \frac{1}{n} \operatorname{Log} z$$

Let $z = re^{j\theta}$

- $\sqrt[n]{z} = \sqrt[n]{r} e^{j\frac{\theta+2k\pi}{n}}$, $k = 0, 1, \ldots, n-1$
- $\operatorname{Log} \sqrt[n]{z} = \frac{1}{n} \log r + j\frac{\theta + 2k\pi}{n} + j2m\pi, \quad k = 0, \ldots, n-1; m \in \mathbb{Z}$
- $\left\{ \frac{k}{n} + m : k = 0, \ldots, n-1; m \in \mathbb{Z} \right\} = \left\{ \frac{k}{n} : k \in \mathbb{Z} \right\}$
- $\operatorname{Log} \sqrt[n]{z} = \frac{1}{n} \log r + j\frac{\theta + 2k\pi}{n}, \quad k \in \mathbb{Z}$
- $\operatorname{Log} z = \log r + j(\theta + 2k\pi), \quad k \in \mathbb{Z}$
- $\operatorname{Log} \sqrt[n]{z} = \frac{1}{n} \operatorname{Log} z$

Logarithm

Consider the principal branch $\log z = \log|z| + j\arg z$

- $\arg z$ is discontinuous on the negative real axis

$$\lim_{y\downarrow 0} \arg(x+iy) = \pi, \quad \lim_{y\uparrow 0} \arg(x+iy) = -\pi$$

- $\log z$ is continuous on $D = \mathbb{C} \setminus (-\infty, 0] = \{z : z \neq 0, |\arg z| < \pi\}$
- For $z \in D$, $\log z \in E = \{w : |\mathrm{Im}\, w| < \pi\}$. Its inverse $z = e^w$ is single valued on E, so

$$(\log z)' = \frac{1}{(e^w)'|_{w=\log z}} = \frac{1}{e^{\log z}} = \frac{1}{z}$$

- Thus $\log z$ is analytic on D with derivative $1/z$.

The other branches $(\log z)_k = \log z + j2k\pi$ are also analytic on D with derivative $(\log z)'_k = (\log z + j2k\pi)' = 1/z$.

Power Function

For $\alpha \in \mathbb{C}$, $z \in \mathbb{C} \setminus \{0\}$, define z^α by

$$z^\alpha = e^{\alpha \operatorname{Log} z}$$

Since $\operatorname{Log} z$ is multivalued, z^α is in general also multivalued. We can define branches of z^α by

$$(z^\alpha)_k = e^{\alpha (\log z)_k}, \qquad \text{where } (\log z)_k = \log z + j2k\pi$$

- $(z^\alpha)_0 = e^{\alpha \log z}$ is called the principal branch of z^α. Note

$$(z^\alpha)_k = e^{j\alpha 2k\pi} (z^\alpha)_0$$

- Depending on α, different k may yield identical $(z^\alpha)_k$
  - If $n \in \mathbb{Z}$, z^n is single valued. Only one branch $(z^n)_k = (z^n)_0$
  - If $\alpha = m/n \in \mathbb{Q}$, where $\gcd(m, n) = 1$, z^α is n-valued, i.e. there are n branches, $(z^\alpha)_{k_1} = (z^\alpha)_{k_2} \iff k_1 - k_2 \in n\mathbb{Z}$
  - If $\alpha \in \mathbb{C} \setminus \mathbb{Q}$, z^α has infinitely many values. Infinitely many branches: $(z^\alpha)_k$ are all different for different $k \in \mathbb{Z}$

Power Function

Example. $1^{\sqrt{2}} = e^{\sqrt{2}\,\mathrm{Log}\,1} = e^{j2\sqrt{2}k\pi}$, $k \in \mathbb{Z}$.

Example. $j^j = e^{j\,\mathrm{Log}\,j} = e^{j(j\frac{\pi}{2}+j2k\pi)} = e^{-(\frac{\pi}{2}+2k\pi)}$, $k \in \mathbb{Z}$.

Analyticity of z^α

- Since $\log z$ is analytic on $D = \mathbb{C} \setminus (-\infty, 0]$ and $\exp$ is analytic on $\mathbb{C}$, the chain rule yields

$$(z^\alpha)_0' = (e^{\alpha \log z})' = e^{\alpha \log z}\frac{\alpha}{z} = \alpha e^{(\alpha-1)\log z} = \alpha (z^{\alpha-1})_0$$

  The principal branch $(z^\alpha)_0$ is analytic on D.
- Other branches are also analytic on D with $(z^\alpha)_k' = \alpha(z^{\alpha-1})_k$.
  - Recall $(z^\alpha)_k = e^{j\alpha 2k\pi}(z^\alpha)_0$
  - $(z^\alpha)_k' = e^{j\alpha 2k\pi}(z^\alpha)_0' = e^{j\alpha 2k\pi}\alpha(z^{\alpha-1})_0 = \alpha(z^{\alpha-1})_k$