EI331 Signals and Systems

Lecture 23

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May 16, 2019

Contents

Integration

Recall for $f(t) = u(t) + jv(t)$ of a real variable t,

$$\int_a^b f(t)dt = \int_a^b u(t)dt + j\int_a^b v(t)dt$$

Given a smooth curve γ parameterized by $z : [a, b] \to \mathbb{C}$, and a function f continuous on γ, the (contour) integral of f along γ is

$$\int_\gamma f(z)dz \triangleq \int_a^b f(z(t))z'(t)dt$$

$\int_\gamma f(z)dz$ is independent of reparametrization.

Proof. If $\tilde{z} : [c, d] \to \mathbb{C}$ is a reparametrization of γ obtained from z by $\tilde{z}(\tau) = z(t(\tau))$, where $t = t(\tau)$ is continuously differentiable and $t'(\tau) > 0$, then the change-of-variable formula yields

$$\int_a^b f(z(t))z'(t)dt = \int_c^d f(z(t(\tau)))z'(t(\tau))t'(\tau)d\tau = \int_c^d f(\tilde{z}(\tau))\tilde{z}'(\tau)d\tau$$

Integration

The integral along a curve is essentially two line integrals of the second kind, which depends on the orientation of the curve,

$$\begin{aligned}\int_\gamma f(z)dz &= \int_a^b [u(x(t),y(t)) + jv(x(t),y(t))] \cdot [x'(t) + jy'(t)]dt \\ &= \int_\gamma (udx - vdy) + j\int_\gamma (udy + vdx)\end{aligned}$$

Let $-\gamma$ (also γ^-) be the opposite of γ parametrized by $\tilde{z} : [-b, -a] \to \mathbb{C}$, where $\tilde{z}(t) = z(-t)$. By a change of variable,

$$\int_{-\gamma} f(z)dz = \int_{-b}^{-a} f(\tilde{z}(t))\tilde{z}'(t)dt = \int_b^a f(z(t))z'(t)dt = -\int_\gamma f(z)dz$$

The positive orientation of a Jordan curve is such that when traveling along it the interior of the curve is always to the left.

Integration

Subdivision of the curve γ

If γ is divided into a finite number of segments, denoted by $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$, then

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \cdots + \int_{\gamma_n} f(z)dz$$

If γ is piecewise smooth, we simply define the integral along γ to be the sum of the integrals along each smooth segment.

NB. We only consider piecewise smooth curves in this course.

Linearity in the integrand

For $a, b \in \mathbb{C}$,

$$\int_\gamma [af(z) + bg(z)]dz = a\int_\gamma f(z)dz + b\int_\gamma g(z)dz$$

Integration

If $|f(z)| \leq M$ on γ, then $\left|\int_\gamma f(z)dz\right| \leq \int_\gamma |f(z)|ds \leq ML_\gamma$, where ds is the infinitesimal arc length, and L_γ is the length of γ.

Proof. Let $z : [a, b] \to \mathbb{C}$ be a parametrization of γ.

$$\left|\int_\gamma f(z)dz\right| = \left|\int_a^b f(z(t))z'(t)dt\right| \leq \int_a^b |f(z(t))| \cdot |z'(t)|dt$$

Recall $ds = \sqrt{|x'(t)|^2 + |y'(t)|^2}dt = |z'(t)|dt$, so

$$\int_a^b |f(z(t))| \cdot |z'(t)|dt = \int_\gamma |f(z)|ds.$$

Since $|f(z)| \leq M$ on γ,

$$\int_a^b |f(z(t))| \cdot |z'(t)|dt \leq M \int_a^b |z'(t)|dt = M \int_\gamma ds = ML_\gamma.$$

Examples

Example. Let $z : [a, b] \to \mathbb{C}$ be a parameterization of a smooth curve γ.

$$\int_\gamma dz = \int_a^b z'(t)dt = z(t)|_{t=a}^{b} = z(b) - z(a)$$

$$\int_\gamma zdz = \int_a^b z(t)z'(t)dt = \frac{1}{2}z^2(t)\Big|_{t=a}^{b} = \frac{1}{2}z^2(b) - \frac{1}{2}z^2(a)$$

NB. The values of the integrals depend only on the endpoints.

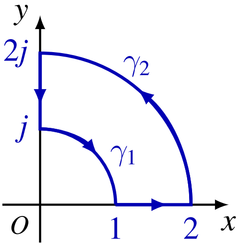
Example. Let $C = \{z \in \mathbb{C} : |z - z_0| = R\}$ be a circle parameterized by $z(t) = z_0 + Re^{jt}$, $t \in [0, 2\pi]$. For $n \in \mathbb{Z}$,

$$\int_C \frac{dz}{(z - z_0)^n} = \int_0^{2\pi} \frac{Rje^{jt}}{(Re^{jt})^n}dt = jR^{1-n} \int_0^{2\pi} e^{j(n-1)t}dt = j2\pi\delta[n - 1]$$

NB. The value is independent of z_0 and R.

Example

Let γ be the boundary of the annulus $\{z : 1 \leq |z| \leq 2\}$ in the first quadrant with positive orientation.



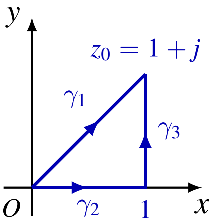
$$\begin{aligned}
\int_\gamma \frac{dz}{\bar{z}} &= \int_1^2 \frac{dx}{x} + \int_{\gamma_2} \frac{zdz}{|z|^2} + \int_2^1 \frac{jdy}{-jy} + \int_{\gamma_1} \frac{zdz}{|z|^2} \\
&= \log 2 + \frac{1}{4}\int_{\gamma_2} zdz + \log 2 + \int_{\gamma_1} zdz \\
&\overset{(*)}{=} \log 2 + \frac{1}{8}(4j^2 - 4) + \log 2 + \frac{1}{2}(1 - j^2) \\
&= 2\log 2
\end{aligned}$$

In $(*)$, we have used the first example on the previous slide.

Example

γ_1 has parameterization $z_1(t) = (1+j)t$, $t \in [0,1]$

$$\int_{\gamma_1} \bar{z} dz = \int_0^1 (1-j)t(1+j)dt = \int_0^1 2tdt = 1$$



γ_2 has parameterization $z_2(t) = t$, $t \in [0,1]$
γ_3 has parameterization $z_3(t) = 1 + jt$, $t \in [0,1]$

$$\begin{aligned}\int_{\gamma_2+\gamma_3} \bar{z} dz &= \int_{\gamma_2} \bar{z} dz + \int_{\gamma_3} \bar{z} dz \\ &= \int_0^1 tdt + \int_0^1 (1-jt)jdt \\ &= \frac{1}{2} + \left(\frac{1}{2} + j\right) = 1 + j\end{aligned}$$

The value of the integral depends not only on the endpoints but also on the path.

Contents

Cauchy's Integral Theorem

Suppose f is analytic in a simply connected domain D and γ a piecewise smooth Jordan curve in D. Recall

$$\int_\gamma f(z)dz = \int_\gamma (udx - vdy) + j\int_\gamma (udy + vdx)$$

If u, v are continuously differentiable on D, then Green's Theorem and the Cauchy-Riemann equations imply

$$\int_\gamma (udx - vdy) = \int_\Omega (-v_x - u_y)dxdy = 0$$

$$\int_\gamma (udy + vdx) = \int_\Omega (u_x - v_y)dxdy = 0$$

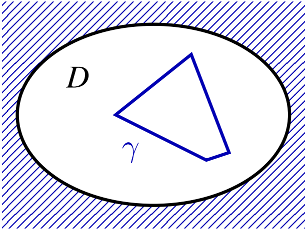
where Ω is the region bounded by γ

Cauchy's Integral Theorem asserts that $\int_\gamma f(z)dz = 0$ without explicitly assuming continuous differentiability of u and v.

Cauchy's Integral Theorem

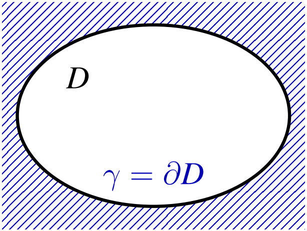
Theorem. If $f(z)$ is analytic in a **simply connected** domain D, and γ is a piecewise smooth closed (possibly not simple) curve in D, then

$$\oint_\gamma f(z)dz = \int_\gamma f(z)dz = 0.$$



Theorem. Let D be the interior of a piecewise smooth Jordan curve γ and $\bar{D} = D \cup \gamma$ its closure. If $f(z)$ is analytic on D and continuous on $\bar{D}$, then

$$\oint_\gamma f(z)dz = \int_\gamma f(z)dz = 0.$$



Cauchy's Integral Theorem

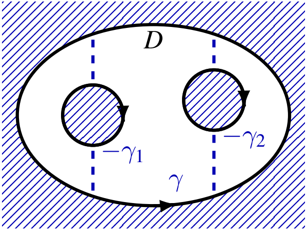
Theorem. Let γ_0, $\gamma_1, \gamma_2, \ldots, \gamma_n$ be $n+1$ positively oriented piecewise smooth Jordan curves such that

(a) $\gamma_1, \ldots, \gamma_n$ lie in the interior of γ_0

(b) each of $\gamma_1, \ldots, \gamma_n$ lies in the exteriors of the others

Let D be the multiply connected domain with boundary $\gamma_0, \gamma_1, \ldots, \gamma_n$. If $f(z)$ is analytic on D and continuous on $\bar{D}$, then

$$\int_{\gamma} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \cdots + \int_{\gamma_n} f(z)dz$$

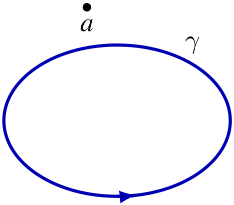


Example

Compute $\int_\gamma \frac{dz}{z-a}$, where γ is a piecewise smooth Jordan curve and $a \notin \gamma$.

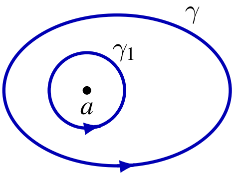
1. $\frac{1}{z-a}$ is analytic on $\mathbb{C} \setminus \{a\}$
2. If a is in the exterior of γ, then

$$\int_\gamma \frac{dz}{z-a} = 0$$



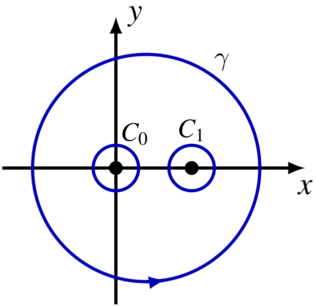
3. If a is in the interior of γ, pick a small enough circle γ_1 centered at a that lies in the interior of γ

$$\int_\gamma \frac{1}{z-a} dz = \int_{\gamma_1} \frac{1}{z-a} dz = j2\pi$$



Example

Let $f(z) = \dfrac{2z-1}{z^2-z}$ and γ be any piecewise smooth Jordan curve containing $|z| = 1$.



1. $\displaystyle\int_\gamma f(z)dz = \int_\gamma \frac{1}{z-1}dz + \int_\gamma \frac{1}{z}dz$
2. $0, 1$ lie in the interior of γ
3. $\displaystyle\int_\gamma \frac{1}{z-1}dz = j2\pi, \qquad \int_\gamma \frac{1}{z}dz = j2\pi$
4. $\displaystyle\int_\gamma f(z)dz = j4\pi$

If $a, b \notin \gamma$, $\operatorname{int}\gamma$ and $\operatorname{ext}\gamma$ are the interior and exterior of γ,

$$\frac{1}{j2\pi}\int_\gamma \frac{dz}{(z-a)(z-b)} = \begin{cases} 0, & \text{if } a, b \in \operatorname{int}\gamma \text{ or } a, b \in \operatorname{ext}\gamma \\ \frac{1}{a-b}, & \text{if } a \in \operatorname{int}\gamma,\ b \in \operatorname{ext}\gamma \\ \frac{1}{b-a}, & \text{if } b \in \operatorname{int}\gamma,\ a \in \operatorname{ext}\gamma \end{cases}$$

Path Independence

Theorem. If $f(z)$ is analytic on a simply connected domain D, then for any piecewise smooth curve γ in D, the integral $\int_\gamma f(z)dz$ depends only on the endpoints of γ.

Proof. Let $z_0, z_1 \in \mathcal{D}$ and γ_1, γ_2 two piecewise smooth curves in D. Then $\gamma = \gamma_1 + \gamma_2^-$ is a closed curve in D. By Cauchy's Theorem,

$$\int_{\gamma_1} f(z)dz + \int_{\gamma_2^-} f(z)dz = \int_\gamma f(z) = 0$$

so

$$\int_{\gamma_1} f(z)dz = -\int_{\gamma_2^-} f(z)dz = \int_{\gamma_2} f(z)dz \triangleq \int_{z_0}^{z_1} f(z)dz$$

If we fix z_0 and let z_1 vary, we can define a function

$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta$$

Primitive

Theorem. If $f(z)$ is analytic on a simply connected domain D and $z_0 \in D$, then the function $F(z) = \int_{z_0}^{z} f(\zeta)d\zeta$ is analytic on D and $F'(z) = f(z)$.

NB. As in calculus, a function F satisfying $F'(z) = f(z)$ is called a primitive of f.

Proof. Fix an arbitrary $z \in D$. When evaluating $\int_{z_0}^{z+\Delta z} f(\zeta)d\zeta$, we can pick a path $z_0 \to z \to z + \Delta z$. Then

$$F(z+\Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(\zeta)d\zeta + \int_{z_0}^{z} f(\zeta)d\zeta = \int_{z}^{z+\Delta z} f(\zeta)d\zeta$$

Since f is analytic, it is also continuous. Given any $\epsilon > 0$, there is a $\delta > 0$ s.t. $|f(\zeta) - f(z)| < \epsilon$ when $|\zeta - z| < \delta$, and $B(z,\delta) \subset D$.

When $|\Delta z| < \delta$, the line segment connecting z and $z + \Delta z$ is contained in $B(z,\delta)$.

Primitive

Proof (cont'd). Note

$$f(z)\Delta z = \int_z^{z+\Delta z} f(z) d\zeta$$

Thus

$$\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_z^{z+\Delta z} [f(\zeta)-f(z)] d\zeta$$

and

$$\left|\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)\right| \leq \frac{1}{|\Delta z|}\int_z^{z+\Delta z} |f(\zeta)-f(z)| ds \leq \epsilon$$

This shows $F'(z) = f(z)$. Since z is arbitrary, F is analytic on D.

Primitive

If F_1, F_2 are two primitives of f, then $F_1 - F_2 = c$ for some constant $c \in \mathbb{C}$.

Theorem. If $f(z)$ is analytic on a simply connected domain D and $z_0 \in D$, and F is a primitive of f, then

$$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0)$$

Proof. Since $\int_{z_0}^{z} f(z)dz$ is primitive of f.

$$\int_{z_0}^{z} f(z)dz = F(z) + c$$

Set $z = z_0$ and use Cauchy's Theorem, $0 = F(z_0) + c$. Thus

$$\int_{z_0}^{z_1} f(z)dz = F(z_1) + c = F(z_1) - F(z_0)$$

Examples

Example. Evaluate $\int_0^j z\cos z dz$.

$z\cos z$ is analytic on $\mathbb{C}$, with a primitive $F(z) = z\sin z + \cos z$.

$$\int_0^j z\cos z dz = F(j) - F(0) = j\sin j + \cos j - 1 = e^{-1} - 1$$

Example. Evaluate $\int_1^j \frac{\log(z+1)}{z+1} dz$ along the arc of the circle $|z| = 1$ in the first quadrant. $\log$ is the principal branch.

$\frac{\log(z+1)}{z+1}$ is analytic in the first quadrant, with a primitive $F(z) = \frac{1}{2}\log^2(z+1)$

$$\int_1^j \frac{\log(z+1)}{z+1} dz = F(j) - F(1) = -\frac{\pi^2}{32} - \frac{3}{8}\log^2 2 + j\frac{\pi\log 2}{8}$$

Contents

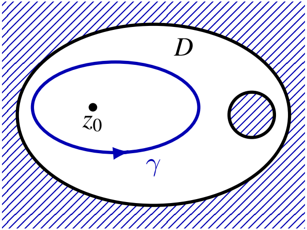
Cauchy's Integral Formula

Theorem. Assume

(a) $f(z)$ is analytic on a domain D

(b) γ is a positively oriented piecewise smooth Jordan curve whose interior lies entirely in D

(c) z_0 is a point in the interior of γ



Then

$$f(z_0) = \frac{1}{j2\pi} \int_\gamma \frac{f(z)}{z - z_0} dz$$

Example.

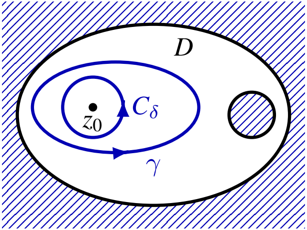
$$\frac{1}{j2\pi} \int_{|z|=4} \frac{\sin z}{z} dz = \sin z \Big|_{z=0} = 0$$

Example.

$$\int_{|z|=4} \left(\frac{1}{z+1} + \frac{2}{z-3} \right) dz = \int_{|z|=4} \frac{dz}{z+1} + \int_{|z|=4} \frac{2dz}{z-3} 2 = j6\pi$$

Proof of Cauchy's Integral Formula

Let C_δ be a circle of radius δ centered at z_0 s.t. $\bar{B}(z_0, \delta)$ is in the interior of γ. Since $\frac{f(z)}{z-z_0}$ is analytic in the domain D_1 bounded by γ and C_δ and continuous on $\bar{D}_1$, Cauchy's Theorem implies



$$\int_\gamma \frac{f(z)}{z-z_0} dz = \int_{C_\delta} \frac{f(z)}{z-z_0} dz$$

Since f is analytic, if δ is small enough, then for any $z \in B(z_0, \delta)$,

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| \leq M \triangleq 2|f'(z_0)|$$

so

$$\left| \int_\gamma \frac{f(z)}{z-z_0} dz - j2\pi f(z_0) \right| = \left| \int_{C_\delta} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq 2\pi\delta M$$

Letting $\delta \to 0$ yields the desired result.