EI331 Signals and Systems

Lecture 24

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Contents

Derivatives of Cauchy-type Integral

Theorem. Assume

(a) γ is a piecewise smooth simple (or Jordan) curve

(b) f is continuous on γ

Then the function defined by the following Cauchy-type integral

$$F(z) \triangleq \frac{1}{j2\pi} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \notin \gamma$$

is analytic on $\mathbb{C} \setminus \gamma$. Moreover, it is infinitely differentiable and all its derivatives are analytic on $\mathbb{C} \setminus \gamma$ with

$$F^{(n)}(z) = \frac{n!}{j2\pi} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta, \quad n = 1, 2, \ldots$$

NB. The formula for $F^{(n)}$ can be obtained by differentiation under the integral sign.

Derivatives of Cauchy-type Integral

Proof. If γ is a simple curve, then $\mathbb{C} \setminus \gamma$ is a domain. If γ is a Jordan curve, then $\mathbb{C} \setminus \gamma$ is the union of two domains by the Jordan Curve Theorem.

(1). We first show $F(z)$ is continuous on $\mathbb{C} \setminus \gamma$.

- For $z_0 \in \mathbb{C} \setminus \gamma$, there exists an open disk $B(z_0, \delta) \subset \mathbb{C} \setminus \gamma$, so $|\zeta - z_0| \geq \delta$ for $\zeta \in \gamma$.
- Let $z \in B(z_0, \delta/2)$. The triangle inequality yields $|z - \zeta| \geq |z_0 - \zeta| - |z - z_0| \geq \delta/2$ for $\zeta \in \gamma$.
- By the definition of F,

$$F(z) - F(z_0) = \frac{z - z_0}{j2\pi} \int_\gamma \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)} d\zeta \qquad (\star)$$

so

$$|F(z) - F(z_0)| \leq \frac{|z - z_0|}{2\pi} \int_\gamma \frac{|f(\zeta)|}{(\delta/2)\delta} |d\zeta| = \frac{|z - z_0|}{\pi\delta^2} \int_\gamma |f(\zeta)| d\zeta$$

Derivatives of Cauchy-type Integral

Proof (cont'd). (2). Next we show

$$F'(z) = \frac{1}{j2\pi}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}d\zeta, \quad \forall z \in \mathbb{C}\setminus\gamma.$$

- By ($\star$) of the previous slide,

$$\frac{F(z)-F(z_0)}{z-z_0} = \frac{1}{j2\pi}\int_\gamma \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)}d\zeta = G(z),$$

  where $g(\zeta) \triangleq \frac{f(\zeta)}{\zeta - z_0}$ and

$$G(z) \triangleq \frac{1}{j2\pi}\int_\gamma \frac{g(\zeta)}{\zeta - z}d\zeta$$

- g is continuous on γ. By (1), G is continuous on $\mathbb{C}\setminus\gamma$, so

$$F'(z_0) = \lim_{z\to z_0}\frac{F(z)-F(z_0)}{z-z_0} = G(z_0) = \frac{1}{j2\pi}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^2}d\zeta$$

Derivatives of Cauchy-type Integral

Proof (cont'd). (3). Finally we show the formula for higher order derivatives by induction.

- Assume the formula holds for $1 \le k \le n$, $n \ge 1$,

$$F^{(k)}(z) = \frac{k!}{j2\pi} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta. \qquad (\dagger)$$

- For $z_0 \in \mathbb{C} \setminus \gamma$, and g, G defined on the previous slide,

$$\begin{aligned} F^{(n)}(z) &= \frac{n!}{j2\pi} \int_\gamma \frac{(\zeta - z_0)g(\zeta)}{(\zeta - z)^{n+1}} d\zeta \\ &= \frac{n!}{j2\pi} \int_\gamma \frac{g(\zeta)}{(\zeta - z)^{n}} d\zeta + \frac{n!}{j2\pi} \int_\gamma \frac{(z - z_0)g(\zeta)}{(\zeta - z)^{n+1}} d\zeta \end{aligned}$$

- Since g is continuous on γ, $(\dagger)$ holds with f replaced by g,

$$F^{(n)}(z) = nG^{(n-1)}(z) + (z - z_0)G^{(n)}(z)$$

Derivatives of Cauchy-type Integral

Proof (cont'd).

- As in (1), let $B(z_0, \delta) \subset \mathbb{C} \setminus \gamma$. $G^{(n)}$ is bounded on $B(z_0, \delta/2)$

$$|G^{(n)}(z)| \leq \frac{n!}{j2\pi} \int_\gamma \frac{|g(\zeta)|}{|\zeta - z|^{n+1}} |d\zeta| \leq \frac{n!}{j2\pi} \int_\gamma \frac{|g(\zeta)|}{(\delta/2)^{n+1}} |d\zeta|$$

- $G^{(n-1)}(z)$ is differentiable and hence continuous

$$\lim_{z \to z_0} F^{(n)}(z) = \lim_{z \to z_0} [nG^{(n-1)}(z) + (z - z_0)G^{(n)}(z)] = F^{(n)}(z_0)$$

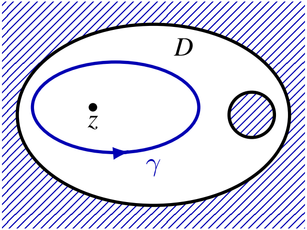
so $F^{(n)}$ is continuous. Similarly, $G^{(n)}$ is continuous

- Let $z \to z_0$,

$$\begin{aligned} \frac{F^{(n)}(z) - F^{(n)}(z_0)}{z - z_0} &= n \frac{G^{(n-1)}(z) - G^{(n-1)}(z_0)}{z - z_0} + G^{(n)}(z) \\ &\to (n+1)G^{(n)}(z_0) = F^{(n+1)}(z_0) \end{aligned}$$

Integral Formula for Derivatives of Analytic Functions

Theorem. If f is analytic on a domain D, then its derivative f' is also analytic on D, and



$$f^{(n)}(z) = \frac{n!}{j2\pi} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta,$$

where γ is a positively oriented piecewise smooth Jordan curve encircling z whose interior lies entirely in D.

Proof. By Cauchy's Integral Formula

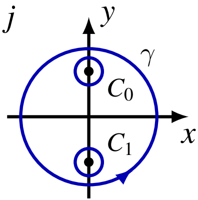
$$f(z) = \frac{1}{j2\pi} \int_{\gamma} \frac{f(z)}{\zeta - z} dz$$

Since f is analytic on D, it is continuous on γ. By the previous theorem, f' is analytic and the derivatives of f are as given.

Example

Let $\gamma = \{z : |z| = 2\}$ be positively oriented and $r > 1$.

Let C_0, C_1 be positively oriented circles centered at j and $-j$ that lie in the interior of γ. By Cauchy's Theorem,



$$\int_\gamma \frac{e^z dz}{(z^2+1)^2} = \int_{C_0} \frac{e^z dz}{(z^2+1)^2} + \int_{C_1} \frac{e^z dz}{(z^2+1)^2}$$

$$\int_{C_0} \frac{e^z dz}{(z^2+1)^2} = \int_{C_0} \frac{\frac{e^z}{(z+j)^2} dz}{(z-j)^2} = j2\pi \left[\frac{e^z}{(z+j)^2} dz\right]'_{z=j} = \frac{(1-j)e^j}{2}\pi$$

$$\int_{C_1} \frac{e^z dz}{(z^2+1)^2} = \int_{C_1} \frac{\frac{e^z}{(z-j)^2} dz}{(z+j)^2} = j2\pi \left[\frac{e^z}{(z-j)^2} dz\right]'_{z=-j} = \frac{-(1+j)e^{-j}}{2}\pi$$

$$\int_\gamma \frac{e^z dz}{(z^2+1)^2} = j\pi \mathrm{Im}\left[(1-j)e^j\right] = j\pi(\sin 1 - \cos 1)$$

Morera's Theorem

Theorem. If f is continuous on a domain D and $\int_\gamma f(z)dz = 0$ for any piecewise smooth Jordan curve γ in D whose interior also lies in D, then f is analytic on D.

Proof. Fix $z_0 \in D$ and an open disk $B(z_0, \delta) \subset D$. It suffices to show f is analytic on $B(z_0, \delta)$.

1. Because $\int_\gamma f(z)dz = 0$ for any piecewise Jordan curves, the integral $\int_{z_0}^z f(z)dz$ is independent of the path in $B(z_0, \delta)$ that connects z_0 and z. Define

$$F(z) = \int_{z_0}^z f(z)dz, \quad z \in B(z_0, \delta).$$

2. Since f is continuous, the proof on slides 16-17 of Lecture 23 shows $F'(z) = f(z)$, so F is analytic on $B(z_0, \delta)$.

3. Thus $f = F'$ is also analytic on $B(z_0, \delta)$.

Cauchy's Inequality

Theorem. If f is analytic on the open disk $B(z_0, R)$, and $|f(z)| \leq M$ on $B(z_0, R)$, then

$$|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}, \quad n \in \mathbb{N}.$$

Proof. Let γ be the circle $|z - z_0| = r$ with $r \in (0, R)$. Then

$$f^{(n)}(z_0) = \frac{n!}{j2\pi} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

Thus

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \int_\gamma \frac{|f(\zeta)|}{|\zeta - z_0|^{n+1}} |d\zeta| \leq \frac{n!}{2\pi} \frac{M}{r^{n+1}} 2\pi r = \frac{n!M}{r}$$

Letting $r \to R$,

$$|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}.$$

Liouville's Theorem

A functions that is analytic on $\mathbb{C}$ is called an entire function.

Theorem. If f is entire and bounded, then it is constant.

Proof.

1. Since f is bounded, there exists an M s.t. $|f(z)| \leq M$, $\forall z \in \mathbb{C}$.
2. For any $z_0 \in \mathbb{C}$ and $R > 0$, Cauchy's inequality on $B(z_0, R)$ yields

$$|f'(z_0)| \leq \frac{M}{R}.$$

3. Letting $R \to \infty$,

$$f'(z_0) = 0$$

4. Since z_0 is arbitrary, $f'(z) \equiv 0$ on $\mathbb{C}$, so f is constant.

Fundamental Theorem of Algebra

Theorem. A polynomial P of degree $n \geq 1$, i.e.

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n, \quad a_n \neq 0$$

has exactly n roots in $\mathbb{C}$.

Proof Sketch. First show P has at least one root.

1. If P does not have a root, then $Q(z) = 1/P(z)$ is entire
2. As $|z| \to \infty$, $Q(z) \to 0$, so $\exists R > 0$ s.t. $|Q(z)| \leq 1$ for $|z| > R$
3. Being continuous, Q is bounded on $|z| \leq R$ and hence on $\mathbb{C}$
4. By Liouville's Theorem, Q is constant. So P is also constant, contradiction.

Let z_0 be a root of P. Factor $P(z) = (z - z_0)P_1(z)$, where P_1 has degree $n - 1$. We can repeat this process until P_1 has degree 0.

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Harmonic Functions

A real function $\phi(x, y)$ of two real variables is called a harmonic function on a domain D if it satisfies the Laplace equation on D,

$$\frac{\partial \phi^2}{\partial x^2} + \frac{\partial \phi^2}{\partial y^2} = 0$$

Theorem. If $f(z) = u(x, y) + jv(x, y)$ is analytic on a domain D, then u and v are harmonic functions on D.

Proof. Since f is analytic, the Cauchy-Riemann equations hold

$$u_x = v_y, \quad u_y = -v_x$$

Since f' is analytic, u and v are continuously differentiable, so

$$u_{xx} = v_{yx} = v_{xy} = -u_{yy}$$

So $u_{xx} + u_{yy} = 0$. Similarly, $v_{xx} + v_{yy} = 0$.

Harmonic Conjugate

If $f = u + jv$ is analytic, v is called a harmonic conjugate of u.

Theorem. A harmonic function on a simply connected domain has a harmonic conjugate.

Example. $u(x,y) = y^3 - 3x^2y$ is harmonic on $\mathbb{C}$. For its conjugate,

1. By the Cauchy-Riemann equations
$$v_x = -u_y = -3y^2 + 3x^2, \quad v_y = u_x = -6xy$$
2. Integrate w.r.t. y,
$$v(x,y) = \int v_y dy = \int (-6xy) dy = -3xy^2 + g(x)$$
3. Differentiate w.r.t. x,
$$v_x = -3y^2 + g'(x) = -3y^2 + 3x^2 \implies g(x) = \int 3x^2 dx = x^3 + c$$
4. $v(x,y) = -3xy^2 + x^3 + c$

Mean-value Property

Theorem. If $f(z)$ is analytic on an open disk $B(z_0, R)$, then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{jt})dt, \quad 0 < r < R.$$

i.e. the mean value of an analytic function on a circle is equal to the value at the center.

Proof. Use Cauchy's Integral Formula and the parameterization $z(t) = re^{jt}$, $t \in [0, 2\pi]$.

Theorem. If $u(x, y)$ is harmonic on an open disk $B(z_0, R)$, then

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{jt})dt$$

Proof. Use the previous theorem and the fact that u is the real part of an analytic function.

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Series of Functions

Recall a numerical series $\sum_{n=1}^{\infty} z_n$ converges to s if the sequence of its partial sums $s_k = \sum_{n=1}^{k} z_n$ converges to s.

Given a sequence of functions $f_n(z)$, $n = 1, 2, \ldots$ defined on a set $\Omega \subset \mathbb{C}$, the infinite series $\sum_{n=1}^{\infty} f_n(z)$ converges to $s(z)$ on Ω, if its partial sum $s_k(z) = \sum_{n=1}^{k} f_n(z)$ converges $s(z)$ at every $z \in \Omega$, i.e.

$$\lim_{k\to\infty} |s_k(z) - s(z)| = 0, \quad \forall z \in \Omega.$$

The function $s(z)$ is called the sum of the series.

Power Series

If $f_n(z) = c_n(z-z_0)^n$, the infinite series $\displaystyle\sum_{n=1}^{\infty} f_n(z) = \sum_{n=1}^{\infty} c_n(z-z_0)^n$ is called a power series.

By a change of variable, we can focus on the case $z_0 = 0$.

Theorem (Abel). If the series $\displaystyle\sum_{n=1}^{\infty} c_n z^n$ converges at $z_0 \neq 0$, then it converges absolutely on the open disk $|z| < |z_0|$. If the series diverges at z_0, then it diverges for $|z| > |z_0|$.

Proof. If the series converges at z_0, then $c_n z_0^n \to 0$ as $n \to \infty$, so it is bounded, i.e. $|c_n z_0^n| \leq M$ for some $M > 0$. For $|z| < |z_0|$, let $q = |z|/|z_0| < 1$. Then $|c_n z^n| = |c_n z_0^n| q^n < M q^n$. Since $\sum_n M q^n$ converges, so does $\sum_n |c_n z^n|$.

If the series diverges at z_0, then it diverges for $|z| > |z_0|$; otherwise, it would contradict what has just been proven.

Power Series

Theorem. $\sum_{n=1}^{\infty} c_n z^n$ has a radius of convergence R s.t. the series converges for $|z| < R$ and diverges for $|z| > R$. Moreover,

$$R = \left(\limsup_{n\to\infty} \sqrt[n]{|c_n|} \right)^{-1}$$

NB. If $R = 0$, the series diverges for every $z \neq 0$. If $R = \infty$, the series converges for every $z \in \mathbb{C}$.

NB. As in calculus, the convergence on the circle $|z| = R$ has to be considered case by case.

Proof. If $|z| < R$, then $\limsup_n \sqrt[n]{|c_n z^n|} = \frac{|z|}{R} < 1$. Fix $\rho \in (\frac{|z|}{R}, 1)$. For all large enough n, $\sqrt[n]{|c_n z^n|} \leq \rho \implies |c_n z^n| \leq \rho^n$. Since $\sum_n \rho^n$ is convergent, so are $\sum_n |c_n z^n|$ and $\sum_n c_n z^n$. If $|z| > R$, then $\limsup_n \sqrt[n]{|c_n z^n|} = \frac{|z|}{R} > 1$, so $\lim_n |c_n z^n| \neq 0$ and $\sum_n c_n z^n$ diverges.

Theorem. If $\lim_n \frac{|c_{n+1}|}{|c_n|} = \lambda$ exists, then the radius of convergence is $R = 1/\lambda$.

Examples

Example. For $\sum_{n=1}^{\infty} n^{-3} z^n$, the radius of convergence $R = 1$, since

$$\lim_{n\to\infty} \frac{(n+1)^{-3}}{n^{-3}} = 1$$

On the circle $|z| = 1$, the series is absolutely convergent, since

$$\sum_{n=1}^{\infty} \frac{|z|^n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges.

Example. For $\sum_{n=1}^{\infty} n^{-1} z^n$, the radius of convergence $R = 1$, since

$$\lim_{n\to\infty} \frac{(n+1)^{-1}}{n^{-1}} = 1$$

At $z = 1$, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

At $z = -1$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the Leibniz test

Properties of Power Series

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ have radii of convergence R_f and R_g, respectively, then for $|z| < \min\{R_f, R_g\}$

- $$f(z) \pm g(z) = \sum_{n=0}^{\infty} (a_n \pm b_n) z^n$$
- $$f(z)g(z) = \left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n$$

  Justified by the absolute convergence of the power series.

NB. The series $h(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n$ may have a larger radius of convergence, but the equality $f(z) + g(z) = h(z)$ makes sense only for $|z| < \min\{R_f, R_g\}$.

Example. $f(z) = \sum_{n=0}^{\infty} z^n$, $g(z) = \sum_{n=0}^{\infty} (1 + a^n)^{-1} z^n$ $(0 < a < 1)$, and $h(z) = \sum_{n=0}^{\infty} \frac{a^n}{1+a^n} z^n$. $R_f = R_g = 1$, $R_h = a^{-1} > 1$.

Properties of Power Series

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R_f, and $g(z)$ is analytic on $|z| < R_g$ and $|g(z)| < R_f$ for $|z| < R_g$, then

$$f[g(z)] = \sum_{n=0}^{\infty} a_n [g(z)]^n$$

Example. For $a \neq b$, find c_n s.t. $\dfrac{1}{z-b} = \sum_{n=0}^{\infty} c_n (z-a)^n$

Solution. We know $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$.

$$\frac{1}{z-b} = -\frac{1}{b-a} \cdot \frac{1}{1 - \frac{z-a}{b-a}} = -\frac{1}{b-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{b-a} \right)^n$$

so $c_n = -(b-a)^{n+1}$. The series converges for $|z-a| < |b-a|$.

Properties of Power Series

If $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ has radius of convergence R, then

- f is analytic on $|z - z_0| < R$.
- f can be differentiated term by term on $|z - z_0| < R$, i.e.

$$f'(z) = \sum_{n=1}^{\infty} n c_n (z - z_0)^{n-1}$$

- f can be integrated term by term on $|z - z_0| < R$, i.e.,

$$\int_\gamma f(z)dz = \sum_{n=0}^{\infty} c_n \int_\gamma (z - z_0)^n dz \quad \text{for } \gamma \text{ in } |z - z_0| < R$$

In particular,

$$\int_{z_0}^{z} f(\zeta)d\zeta = \sum_{n=0}^{\infty} \frac{c_n}{n+1} (z - z_0)^{n+1}$$