EI331 Signals and Systems

Lecture 27

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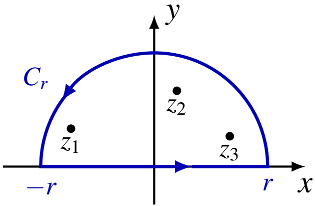
Contents

$\int_{-\infty}^{\infty} R(x)dx$

$R(x) = \frac{N(x)}{D(x)}$ is a rational function of x, where N, D are polynomials with $\deg D \geq \deg N + 2$, and R has no singularity on the real axis.

1. Pick a r large enough s.t. the upper half disk centered at 0 contains all the singularities $z_1, \ldots, z_K$, of $R(z)$ in the upper half plane (we don't care about those in the lower half plane)

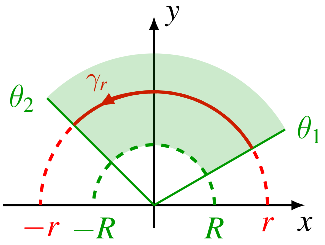


2. $$\int_{-r}^{r} R(x)dx + \int_{C_r} R(z)dz = j2\pi \sum_{k=1}^{K} \mathrm{Res}(R, z_k)$$
3. $\lim_{r\to\infty} \int_{C_r} R(z)dz = 0$ by the condition $\deg D \geq \deg N + 2$
4. $$\int_{-\infty}^{\infty} R(x)dx = j2\pi \sum_{k=1}^{K} \mathrm{Res}(R, z_k)$$

$\int_{-\infty}^{\infty} R(x)dx$

Lemma. Suppose $f(z)$ is continuous on $D = \{z : R < |z| < \infty, \theta_1 \leq \arg z \leq \theta_2\}$, where $0 \leq \theta_1 < \theta_2 \leq 2\pi$. Let γ_r be the arc $z(t) = re^{j\theta}$, $r > R$, $\theta \in [\theta_1, \theta_2]$. If $\limsup_{D \ni z \to \infty} zf(z) = 0$, then

$$\lim_{r\to\infty} \int_{\gamma_r} f(z)dz = 0$$



Proof. Since $\limsup_{D \ni z \to \infty} zf(z) = 0$, given any $\epsilon > 0$, there exists an R_ϵ s.t. $|zf(z)| < \frac{\epsilon}{\theta_2 - \theta_1}$ for $z \in D$ and $|z| > R_0$. For $r > R_0$,

$$\left|\int_{\gamma_r} f(z)dz\right| \leq \int_{\gamma_r} |f(z)|\, ds \leq \int_{\gamma_r} \frac{\epsilon}{(\theta_2 - \theta_1)r} ds = \epsilon$$

NB. Item 3 of the previous slide follows from the lemma with $\theta_1 = 0$ and $\theta_2 = \pi$.

Example

Evaluate $I = \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + a^2)(x^2 + b^2)}$, where $a, b > 0$

Solution. $R(z) = \frac{z^2}{(z^2 + a^2)(z^2 + b^2)}$ has four simple poles at $z = \pm ja, \pm jb$. Two of them, ja, jb, are in the upper half plane. The residues are

$$\mathrm{Res}(R, ja) = \lim_{z \to ja} (z - ja) R(z) = \frac{a}{2j(a^2 - b^2)}$$

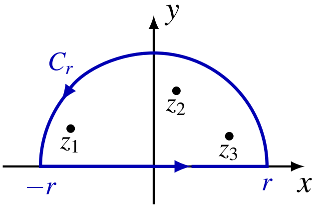
$$\mathrm{Res}(R, jb) = \lim_{z \to jb} (z - jab) R(z) = \frac{-b}{2j(a^2 - b^2)}$$

Thus

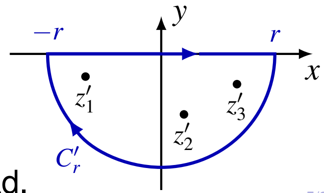
$$I = j2\pi [\mathrm{Res}(R, ja) + \mathrm{Res}(R, jb)] = \frac{\pi}{a + b}$$

$\int_{-\infty}^{\infty} R(x) e^{jax} dx \ (a > 0)$

$R(x) = \frac{N(x)}{D(x)}$ is a rational function of x, where $\deg D \geq \deg N + 1$, and R has no singularity on the real axis.



1. Pick a r large enough s.t. the upper half disk centered at 0 contains all the singularities $z_1, \ldots, z_K$, of $R(z)$ in the upper half plane
2. $$\int_{-r}^{r} R(x) e^{jax} dx + \int_{C_r} R(z) e^{jaz} dz = j2\pi \sum_{k=1}^{K} \text{Res}[R(z) e^{jaz}, z_k]$$
3. $\lim_{r \to \infty} \int_{C_r} R(z) e^{jaz} dz = 0$ by the condition $\deg D \geq \deg N + 1$ and Jordan's Lemma
4. $$\int_{-\infty}^{\infty} R(x) e^{jax} dx = j2\pi \sum_{k=1}^{K} \text{Res}[R(z) e^{jaz}, z_k]$$



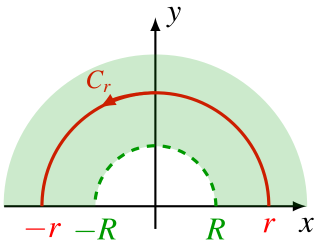
NB. If $a < 0$, use the lower half disk instead.

Jordan's Lemma

Lemma. Suppose $f(z)$ is continuous on $D = \{z : R < |z| < \infty, \operatorname{Im} z \geq 0\}$. If $\lim_{D \ni z \to \infty} f(z) = 0$, and $a > 0$, then

$$\lim_{r \to \infty} \int_{C_r} f(z) e^{jaz} dz = 0$$

where C_r is $z(t) = re^{j\theta}$, $r > R$, $\theta \in [0, \pi]$.



Proof. Let $I_r = \int_{C_r} f(z) e^{jaz} dz$ and $M_r = \max_{z \in C_r} |f(z)|$. Then

$$|I_r| \leq \int_{\gamma_r} |f(z) e^{jaz}| ds \leq M_r \int_0^{\pi} e^{-ar \sin\theta} r d\theta = 2M_r \int_0^{\pi/2} e^{-ar \sin\theta} r d\theta$$

Using $\sin\theta \geq \frac{2}{\pi}\theta$ for $\theta \in [0, \frac{\pi}{2}]$ and $\lim_{r \to \infty} M_r = 0$,

$$|I_r| \leq 2M_r \int_0^{\pi/2} e^{-ar\frac{2}{\pi}\theta} r d\theta = \frac{\pi M_r}{a}(1 - e^{-ar}) \to 0, \text{ as } r \to \infty$$

NB. If $a < 0$, the lemma still holds if we replace D and C_r by $\{z : R < |z| < \infty, \operatorname{Im} z \leq 0\}$ and $z(t) = re^{-j\theta}$, $r > R$, $\theta \in [0, \pi]$.

Example

Evaluate $I = \int_0^\infty \frac{x \sin x}{x^2 + a^2} dx$, where $a > 0$

Solution. $R(z) = \frac{z}{z^2 + a^2}$ has two simple poles at $z = \pm ja$. The pole ja is in the upper half plane,

$$\mathrm{Res}[R(z)e^{jz}, ja] = \lim_{z \to ja} (z - ja) R(z) e^{jz} = \frac{e^{-a}}{2}$$

Thus

$$\int_{-\infty}^{\infty} \frac{x e^{jx}}{x^2 + a^2} dx = j2\pi \, \mathrm{Res}[R(z)e^{jz}, ja] = j\pi e^{-a}$$

Since $\frac{x \sin x}{x^2 + a^2}$ is even,

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \frac{1}{2} \mathrm{Im} \int_{-\infty}^{\infty} \frac{x e^{jx}}{x^2 + a^2} dx = \frac{\pi}{2} e^{-a}$$

Example

Find the inverse CTFT of $X(j\omega) = \frac{1}{a+j\omega}$, where $a > 0$

Solution.

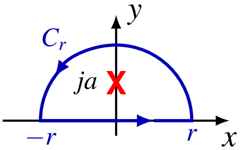
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{a+j\omega} e^{j\omega t} d\omega$$

$$R(z) = \frac{1}{a+jz}$$ has a simple poles at $z = ja$ in the upper half plane.

For $t > 0$,

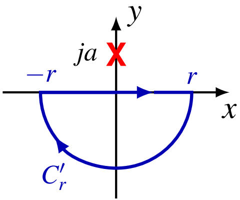
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\omega)e^{j\omega t} d\omega = j\,\mathrm{Res}[R(z)e^{jzt}, ja]$$

$$= j\lim_{z\to ja}(z-ja)R(z)e^{jzt} = e^{-at}$$



For $t < 0$, $R(z)$ is analytic in the lower half plane, so

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\omega)e^{j\omega t} d\omega = 0$$



Therefore, $x(t) = e^{-at}u(t)$.

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$\mathcal{Z}$-transform

Recall the response of a DT LTI system to the input $x[n] = z^n$ is

$$y[n] = (x * h)[n] = H(z)z^n$$

where h is the impulse response of the system and

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

The system function $H(z)$ is called the z-transform of h.

In general, the z-transform of a DT signal $x[n]$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

also denoted by

$$X = \mathcal{Z}\{x\}, \qquad \text{or} \qquad x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$$

$\mathcal{Z}$-transform

Note

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} c_n z^n$$

is a Laurent series at $z = 0$ whose coefficient for z^n is $c_n = x[-n]$.

As a Laurent series, the z-transform converges on an annulus centered at $z = 0$, called its region of convergence (ROC)

Relation with DTFT

$$z = re^{j\omega} \implies X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} \{x[n]r^{-n}\}e^{-j\omega n} = \mathcal{F}\{x[n]r^{-n}\}$$

If the ROC includes the unit circle, setting $r = 1$ yields

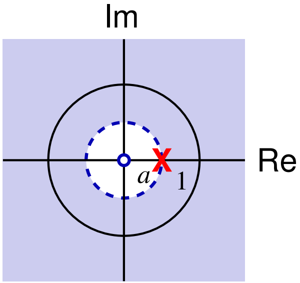
$$X(z)\Big|_{z=e^{j\omega}} = X(e^{j\omega}) = \mathcal{F}\{x\}(e^{j\omega})$$

Example

For $x[n] = a^n u[n]$,

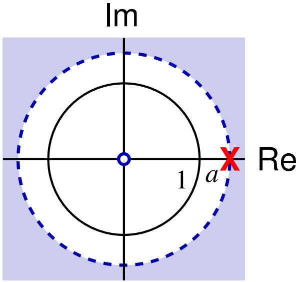
$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a},$$

with ROC $|z| > |a|$.



If $|a| < 1$, the ROC contains the unit circle,

$$\mathcal{F}\{x\}(e^{j\omega}) = X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$



If $|a| > 1$, the DTFT does not exist.

If $|a| = 1$, the DTFT exists as a distribution,

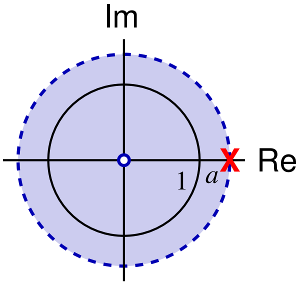
$$\mathcal{F}\{x\}(e^{j\omega}) \neq X(z)\Big|_{z=e^{j\omega}}$$

Example

For $x[n] = -a^n u[-n-1]$,

$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n} = \frac{1}{1 - az^{-1}} = \frac{z}{z-a},$$

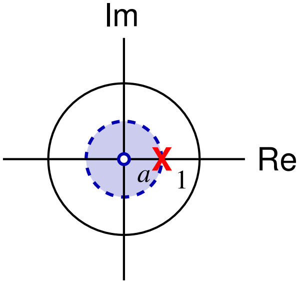
with ROC $|z| < |a|$.



If $|a| > 1$, the ROC contains the unit circle,

$$\mathcal{F}\{x\}(e^{j\omega}) = X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

If $|a| < 1$, the DTFT does not exist.

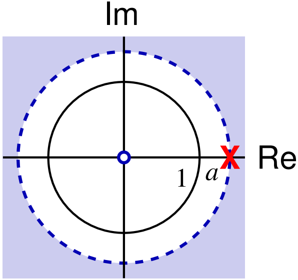


If $|a| = 1$, the DTFT exists as a distribution,

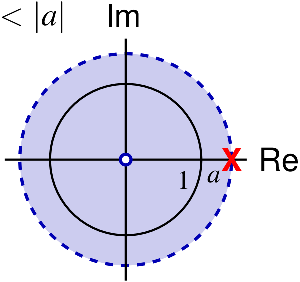
$$\mathcal{F}\{x\}(e^{j\omega}) \neq X(z)\Big|_{z=e^{j\omega}}$$

Importance of ROC

$$x_1[n] = a^n u[n] \stackrel{z}{\longleftrightarrow} X_1(z) = \frac{z}{z-a}, \ |z| > |a|$$



$$x_2[n] = -a^n u[-n-1] \stackrel{z}{\longleftrightarrow} X_2(z) = \frac{z}{z-a}, \ |z| < |a|$$



Different signals can have the same $X(z)$ but different ROCs, consistent with the fact a function has different Laurent series in different annuli of analyticity.

Always specify ROC for z-transforms!

Example

For $x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$,

$$X(z) = \sum_{n=0}^{\infty}\left[7\left(\frac{1}{3}\right)^n - 6\left(\frac{1}{2}\right)^n\right] z^{-n}$$

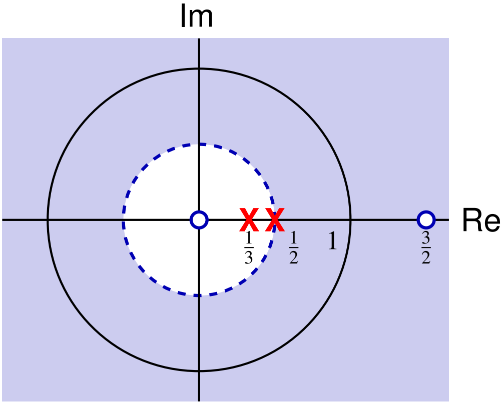
$$= \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}}$$

$$= \frac{1-\frac{3}{2}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})}$$

$$= \frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}$$

with ROC $|z| > \frac{1}{2}$.

$$R = \limsup_{n\to\infty} \sqrt[n]{|x[n]|} = \frac{1}{2}$$



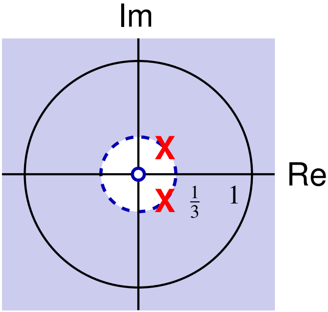
Example

For $x[n] = \left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right) u[n] = \frac{1}{2j}\left(\frac{1}{3}e^{j\frac{\pi}{4}}\right)^n u[n] - \frac{1}{2j}\left(\frac{1}{3}e^{-j\frac{\pi}{4}}\right)^n u[n]$,

$$X(z) = \frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{j\frac{\pi}{4}}z^{-1}} - \frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{-j\frac{\pi}{4}}z^{-1}}$$

$$= \frac{\frac{1}{3\sqrt{2}}z}{(z-\frac{1}{3}e^{j\frac{\pi}{4}})(z-\frac{1}{3}e^{-j\frac{\pi}{4}})}$$

with ROC $|z| > \frac{1}{3}$.



Simple poles at $z = \frac{1}{3}e^{j\frac{\pi}{4}}$ and $z = \frac{1}{3}e^{-j\frac{\pi}{4}}$

A simple zero at $z = 0$

By $\zeta = \frac{1}{z}$, $X(\zeta) = \frac{\frac{1}{3\sqrt{2}}\zeta}{(1-\frac{1}{3}e^{j\frac{\pi}{4}}\zeta)(1-\frac{1}{3}e^{-j\frac{\pi}{4}}\zeta)}$, so $X(z)$ also has a simple zero at ∞.

Rational Transforms

A rational transform X has the following form

$$X(z) = \frac{N(z)}{D(z)}$$

where N, D are polynomials that are coprime, i.e. they have no common factors of degree ≥ 1.

By the Fundamental Theorem of Algebra,

$$X(z) = A \frac{\prod_{k=1}^{n}(z - z_k)}{\prod_{k=1}^{m}(z - p_k)}$$

with the convention $\prod_{k=1}^{0} \cdot = 1$.

- $z_1, \ldots, z_n$ are the finite zeros of X
- $p_1, \ldots, p_m$ are the finite poles of X
- If $n > m$, X has a pole of order $n - m$ at ∞
- If $n < m$, X has a zero of order $m - n$ at ∞

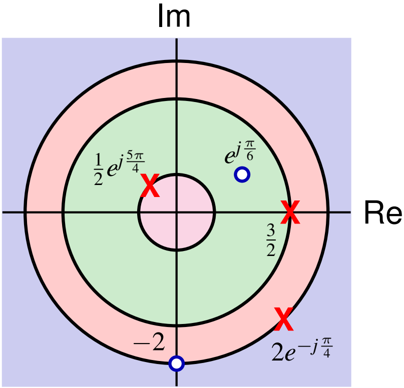
Rational Transforms

A rational function X is determined by its zeros and poles in $\mathbb{C}$, including their orders, up to a multiplicative constant factor.

A rational z-transform is determined by its pole-zero plot and ROC, up to a multiplicative constant factor.

Example.

$$X(z) = A\frac{(z+2)(z-e^{j\frac{\pi}{6}})}{(z-\frac{3}{2})(z-\frac{1}{2}e^{j\frac{5\pi}{4}})(z-2e^{-j\frac{\pi}{4}})}$$



Four possibilities for ROC

- $|z| < 1/2$
- $1/2 < |z| < 3/2$
- $3/2 < |z| < 2$
- $2 < |z| < \infty$

Properties of ROC

- The ROC[1] of $X(z)$ is an annulus $R_1 < |z| < R_2$
- $X(z)$ is analytic in the ROC
- If x is of finite duration, then $R_1 = 0$ and $R_2 = \infty$

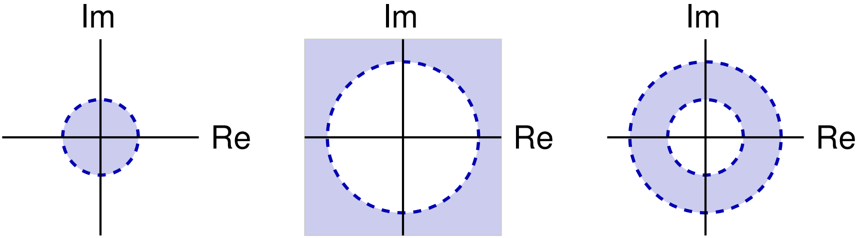
$$X(z) = \sum_{n=N_1}^{N_2} x[n] z^{-n}, \qquad \text{where } x[N_1] \neq 0, x[N_2] \neq 0$$

  - if x causal, i.e. $N_1 \geq 0$, ROC $= \bar{\mathbb{C}} \setminus \{0\}$
  - if x is anticausal, i.e. $N_2 \leq 0$, ROC $= \mathbb{C}$
  - if x is neither causal nor anticausal, ROC = $\mathbb{C} \setminus \{0\}$

[1] X may also converge at some points on the boundary, but for simplicity, we do not include them in the ROC, except for 0 and ∞.

Properties of ROC

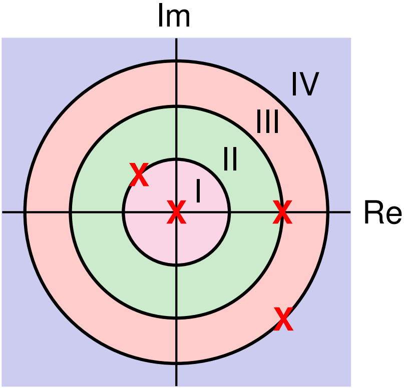
- If $x[n]$ is left-sided, then $R_1 = 0$
- If $x[n]$ is right-sided, then $R_2 = \infty$
- If $x[n]$ is two-sided, then $R_1 > 0$ and $R_2 < \infty$



left-sided right-sided two-sided

Properties of ROC

- If $X(z)$ is rational, the ROC is bounded by poles or extends to 0 (inward) or ∞ (outward).



- If $x[n]$ is also right-sided, then $R_2 = \infty$ and $R_1 = \max_k |p_k|$, where p_k are the poles in $\mathbb{C}$, e.g. region IV

- If $x[n]$ is also left-sided, then $R_1 = 0$ and $R_2 = \min_k |p_k|$, where p_k are the poles in $\mathbb{C} \setminus \{0\}$, e.g. region I

Inverse Z-transform

Recall $x[n]$ is the coefficient of z^{-n} in the Laurent series of $X(z)$,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} c_n z^n$$

For any positively oriented circle C centered at 0 in the ROC,

$$x[n] = c_{-n} = \frac{1}{j2\pi}\int_C \frac{X(z)}{z^{-n+1}}dz = \frac{1}{j2\pi}\int_C X(z)z^{n-1}dz$$

which can be evaluated directly or using the Residue Theorem.

Alternatively, we can use any other method to find the Laurent series expansion of $X(z)$, e.g. partial fraction expansion, and power series expansion for some known functions.

Remember $x[n]$ is the coefficient of z^{-n}, not z^n!

Contents

Properties of $\mathcal{Z}$-transform

Linearity

If

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \quad \text{with ROC}_1$$

$$y[n] \overset{\mathcal{Z}}{\longleftrightarrow} Y(z) \quad \text{with ROC}_2$$

then

$$ax[n] + by[n] \overset{\mathcal{Z}}{\longleftrightarrow} aX(z) + bY(z) \quad \text{with ROC} \supset \text{ROC}_1 \cap \text{ROC}_2$$

Time shifting

If

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z), \quad R_1 < |z| < R_2$$

then

$$x[n - n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0} X(z), \quad R_1 < |z| < R_2$$

NB. The convergence property at 0 and ∞ may change.

Properties of $\mathcal{Z}$-transform

Scaling in the z-domain

If

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z), \quad R_1 < |z| < R_2$$

then for $z_0 \neq 0$,

$$z_0^n x[n] \xleftrightarrow{\mathcal{Z}} X\left(\frac{z}{z_0}\right), \quad |z_0|R_1 < |z| < |z_0|R_2$$

Time reversal

If

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z), \quad R_1 < |z| < R_2$$

then

$$x[-n] \xleftrightarrow{\mathcal{Z}} X(z^{-1}), \quad \frac{1}{R_2} < |z| < \frac{1}{R_1}$$

Properties of $\mathcal{Z}$-transform

Time expansion

Recall the time expansion of x is

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } k \mid n \\ 0, & \text{otherwise} \end{cases}$$

If

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z), \quad R_1 < |z| < R_2$$

then

$$x_{(k)}[n] \xleftrightarrow{\mathcal{Z}} X(z^k), \quad R_1^{1/k} < |z| < R_2^{1/k}$$

Conjugation

If

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z), \quad R_1 < |z| < R_2$$

then

$$x^*[n] \xleftrightarrow{\mathcal{Z}} X^*(z^*), \quad R_1 < |z| < R_2$$

Properties of Z-transform

Differentiation in the z-domain

If

$$x[n] \overset{z}{\longleftrightarrow} X(z), \quad R_1 < |z| < R_2$$

then

$$nx[n] \overset{z}{\longleftrightarrow} -z\frac{d}{dz}X(z), \quad R_1 < |z| < R_2$$

This follows from term-by-term differentiation of Laurent series.

Example.

$$\frac{1}{1-az^{-1}} = \sum_{n=0}^{\infty} a^n z^{-n}, \quad |z| > |a|$$

$$\frac{d}{dz}\frac{1}{1-az^{-1}} = \sum_{n=0}^{\infty} (-n)a^n z^{-n-1}, \quad |z| > |a|$$

$$\frac{az^{-1}}{(1-az^{-1})^2} = -z\frac{d}{dz}\frac{1}{1-az^{-1}} = \sum_{n=0}^{\infty} na^n z^{-n}, \quad |z| > |a|$$

Properties of Z-transform

Initial Value Theorem. If x is causal, i.e. $x[n] = 0$ for $n < 0$, then

$$x[0] = \lim_{z\to\infty} X(z)$$

Proof.

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \sum_{n=0}^{\infty} x[n]z^{-n} = \lim_{\zeta\to 0} \sum_{n=0}^{\infty} x[n]\zeta^{n} \overset{(*)}{=} X_1(0) = x[0]$$

where $(*)$ follows from the continuity of $X_1(\zeta) = \sum_{n=0}^{\infty} x[n]\zeta^n$.

Example. For $x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$,

$$X(z) = \frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}, \quad |z| > \frac{1}{2}$$

We can verify

$$x[0] = 1 = \lim_{z\to\infty} X(z)$$

Properties of Z-transform

Final Value Theorem. If x is causal, and the ROC contains the unit circle, then

$$\lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X(z)$$

Proof.

$$(z-1)X(z) = (z-1)\sum_{n=0}^{\infty} x[n]z^{-n} = zx[0] + \sum_{n=0}^{\infty}(x[n+1]-x[n])z^{-n}$$

Let $z \to 1$,

$$\begin{aligned}\lim_{z\to 1}(z-1)X(z) &\overset{(*)}{=} x[0] + \sum_{n=0}^{\infty}(x[n+1]-x[n]) \\ &= x[0] + \lim_{N\to\infty}\sum_{n=0}^{N-1}(x[n+1]-x[n]) = \lim_{N\to\infty} x[N]\end{aligned}$$

where $(*)$ follows from the uniform convergence of the series.

NB. The condition can be relaxed by Abel's Theorem.

Properties of $\mathcal{Z}$-transform

Convolution property

If

$$x[n] \overset{z}{\longleftrightarrow} X(z) \quad \text{with ROC}_1$$

$$y[n] \overset{z}{\longleftrightarrow} Y(z) \quad \text{with ROC}_2$$

then

$$(x * y)[n] \overset{z}{\longleftrightarrow} X(z)Y(z) \quad \text{with ROC} \supset \text{ROC}_1 \cap \text{ROC}_2$$

Proof.

$$X(z)Y(z) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[n]y[m]z^{-(n+m)} = \sum_{k=-\infty}^{\infty} c[k]z^{-k}$$

Collecting terms of the same power z^{-k}

$$c[k] = \sum_{n+m=k} x[n]y[m] = \sum_{n=-\infty}^{\infty} x[n]y[k-n] = (x * y)[k]$$

Contents

DT System Function

Recall the response of a DT LTI system to the input $x[n]$ is

$$y[n] = (x * h)[n]$$

where h is the impulse response of the system.

If x and h have z-transforms, the convolution property implies

$$Y(z) = X(z)H(z)$$

in their common ROC.

If the ROC has a nonempty interior point, the system function (aka transfer function) $H(z)$ uniquely determines h and hence system properties through the Laurent series expansion

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

Causality

Recall h is right-sided iff the ROC of $H(z)$ is the exterior of a circle, i.e. $R_1 < |z| < \infty$

$$H(z) = \sum_{n=N_1}^{\infty} h[n]z^{-n}$$

The following conditions are equivalent

1. $N_1 \geq 0$
2. $H(z^{-1}) = \sum_{n=N_1}^{\infty} h[n]z^n$ is a convergent power series on $|z| < \frac{1}{R^1}$
3. 0 is a removable singularity of $H(z^{-1})$
4. ∞ is removable singularity of $H(z)$
5. $\lim_{z \to 0} H(z^{-1})$ exists and is finite
6. $\lim_{z \to \infty} H(z)$ exists and is finite[2]

[2] This is the more precise statement of $\infty \in$ ROC.

Causality

An LTI system with system function $H(z)$ is causal iff

1. the ROC is the exterior of a circle
2. $\lim_{z\to\infty} H(z)$ exists and is finite

causal $\iff$ ROC is the exterior of a circle including ∞

An LTI system with rational system function $H(z) = \frac{N(z)}{D(z)}$ is causal iff

1. the ROC is $|z| > |p|$, where p is the outermost pole
2. $\deg D \geq \deg N$

Example. $H(z) = \frac{z^3 - 2z^2 - z}{z^2 + \frac{1}{4}z + \frac{1}{8}}$ cannot be the system function of a causal system.

Stability

Recall an LTI system is stable iff its impulse response $h \in \ell_1$, i.e.

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

i.e. $H(z)$ converges absolutely on the unit circle $|z| = 1$, so its ROC $R_1 < |z| < R_2$ must satisfy $R_1 < 1 < R_2$.

stable $\iff$ ROC includes the unit circle $|z| = 1$

A **causal** LTI system with rational system function $H(z)$ is stable iff all its poles are inside the unit circle.

Example. A causal system with $H(z) = \frac{1}{1-az^{-1}}$ is stable iff $|a| < 1$

Example. A system with $H(z) = \frac{1}{1-az^{-1}}$ where $|a| > 1$ and ROC $|z| < |a|$ is also stable, but it is noncausal.