EI331 Signals and Systems

Lecture 29

Bo Jiang

John Hopcroft Center for Computer Science
Shanghai Jiao Tong University

June 6, 2019

Contents

Laplace Transform

Recall the response of a CT LTI system to the input $x(t) = e^{st}$ is

$$y(t) = (x * h)(t) = H(s)e^{st}$$

where h is the impulse response of the system and

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

The system function $H(s)$ is called the Laplace transform of h.

In general, the Laplace transform of a CT signal $x(t)$ is

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \lim_{\substack{T_1\to\infty \\ T_2\to\infty}} \int_{-T_1}^{T_2} x(t)e^{-st}dt$$

also denoted by

$$X = \mathcal{L}\{x\}, \quad \text{or} \quad x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

Laplace Transform

The set of s for which the integral defining Laplace transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \lim_{\substack{T_1\to\infty \\ T_2\to\infty}} \int_{-T_1}^{T_2} x(t)e^{-st}dt$$

converges is called its region of convergence (ROC)

Relation with CTFT

For $s = \sigma + j\omega$,

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} \{x(t)e^{-\sigma t}\}e^{-j\omega t}dt = \mathcal{F}\{x(t)e^{-\sigma t}\}$$

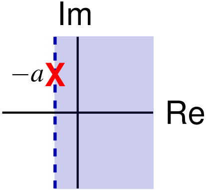
If the ROC includes the imaginary axis, setting $\sigma = 0$ yields

$$X(s)\big|_{s=j\omega} = X(j\omega) = \mathcal{F}\{x\}(j\omega)$$

Example

For $x(t) = e^{-at}u(t)$,

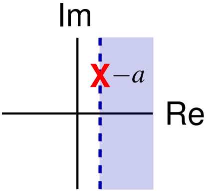
$$X(s) = \int_0^\infty e^{-at}e^{-st}dt = \lim_{T\to\infty} \frac{1 - e^{-(s+a)T}}{s+a} = \frac{1}{s+a},$$



with ROC given by Re $s > -$Re a.

If Re $a > 0$, the ROC contains the imaginary axis,

$$\mathcal{F}\{x\}(j\omega) = X(s)\big|_{z=e^{j\omega}} = \frac{1}{j\omega + a}$$



If Re $a < 0$, the CTFT does not exist.

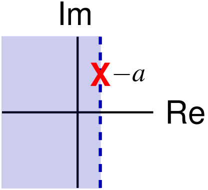
If Re $a = 0$, the CTFT exists only as a distribution,

$$a = j\omega_0 \implies \mathcal{F}\{x\}(e^{j\omega}) = \frac{1}{j(\omega + \omega_0)} + \pi\delta(\omega + \omega_0) \neq X(s)\big|_{s=j\omega}$$

Example

For $x(t) = -e^{-at}u(-t)$,

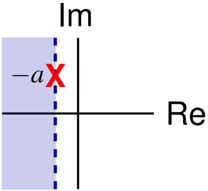
$$X(s) = -\int_{-\infty}^{0} e^{-at}e^{-st}dt = \lim_{T\to\infty} \frac{1-e^{(s+a)T}}{s+a} = \frac{1}{s+a},$$



with ROC given by Re $s < -$Re a.

If Re $a < 0$, the ROC contains the imaginary axis,

$$\mathcal{F}\{x\}(j\omega) = X(s)\Big|_{z=e^{j\omega}} = \frac{1}{j\omega + a}$$



If Re $a > 0$, the CTFT does not exist.

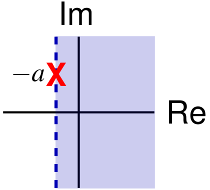
If Re $a = 0$, the CTFT exists only as a distribution,

$$a = j\omega_0 \implies \mathcal{F}\{x\}(e^{j\omega}) = \frac{1}{j(\omega+\omega_0)} + \pi\delta(\omega+\omega_0) \neq X(s)\Big|_{s=j\omega}$$

Importance of ROC

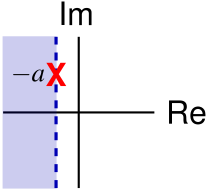
$$x_1(t) = e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s) = \frac{1}{s+a},$$

ROC $\text{Re}\, s > -\text{Re}\, a$



$$x_2(t) = -e^{-at}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s) = \frac{1}{s+a},$$

ROC $\text{Re}\, s < -\text{Re}\, a$



Different signals can have the same $X(s)$ but different ROCs

Always specify ROC for Laplace transforms!

Example

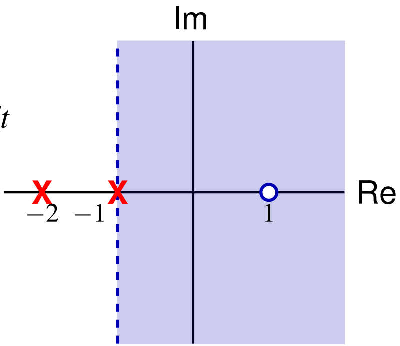
For $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$,

$$X(s) = \int_0^\infty \left[3e^{-2t}u(t) - 2e^{-t}u(t)\right] e^{-st}dt$$

$$= \frac{3}{s+2} - \frac{2}{s+1}$$

$$= \frac{s-1}{(s+2)(s+1)}$$

with ROC Re $s > -1$.



Two simple poles at $s = -2$ and $s = -1$

A simple zero at $s = 1$

Also a simple zero at ∞.

Example

For $x(t) = e^{-2t}u(t) + e^{-t}\cos(3t)u(t) = e^{-2t} + \frac{1}{2}e^{-(1-3j)t} + \frac{1}{2}e^{-(1+3j)t}$,

$$X(s) = \frac{1}{s+2} + \frac{1}{2}\frac{1}{s+(1-3j)} + \frac{1}{2}\frac{1}{s+(1+3j)}$$

$$= \frac{2s^2+5s+12}{(s^2+2s+10)(s+1)}$$

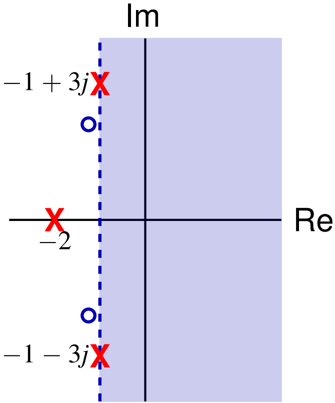
$$= \frac{(s+\frac{5+j\sqrt{71}}{4})(s+\frac{5-j\sqrt{71}}{4})}{(s+2)(s+1-3j)(s+1+3j)}$$

with ROC Re $s > -1$.

Simple poles at $s = -2$ and $s = -1 \pm 3j$

Simple zeros at $s = \frac{-5 \pm j\sqrt{71}}{4}$

Also a simple zero at ∞.



Rational Transforms

A rational transform X has the following form

$$X(s) = \frac{N(s)}{D(s)}$$

where N, D are polynomials that are coprime, i.e. they have no common factors of degree ≥ 1.

By the Fundamental Theorem of Algebra,

$$X(s) = A \frac{\prod_{k=1}^{n}(s - z_k)}{\prod_{k=1}^{m}(s - p_k)}$$

with the convention $\prod_{k=1}^{0} \cdot = 1$.

- $z_1, \ldots, z_n$ are the finite zeros of X
- $p_1, \ldots, p_m$ are the finite poles of X
- If $n > m$, X has a pole of order $n - m$ at ∞
- If $n < m$, X has a zero of order $m - n$ at ∞

Rational Transforms

A rational function X is determined by its zeros and poles in $\mathbb{C}$, including their orders, up to a multiplicative constant factor.

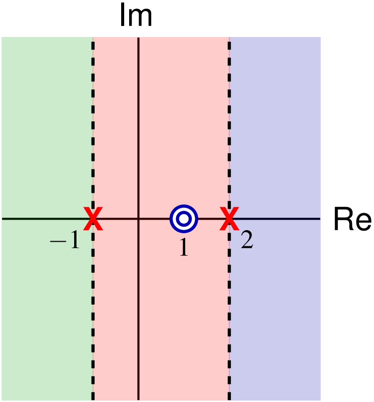
A rational Laplace transform is determined by its pole-zero plot and ROC, up to a multiplicative constant factor.

Example.

$$X(s) = A\frac{(s-1)^2}{(s+1)(s-2)}$$

We will see there are three possibilities for the ROC

- $\text{Re}\, s < -1$
- $-1 < \text{Re}\, s < 2$
- $\text{Re}\, s > 2$



Contents

Convergence of Laplace Transform

Assume $x(t)$ is integrable on any finite interval $[T_1, T_2]$. The convergence of the Laplace transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \lim_{\substack{T_1\to\infty \\ T_2\to\infty}} \int_{-T_1}^{T_2} x(t)e^{-st}dt$$

is equivalent to the convergence of the following two integrals

$$\int_{0}^{\infty} x(t)e^{-st}dt = \lim_{T_2\to\infty} \int_{0}^{T_2} x(t)e^{-st}dt \qquad (\star)$$

$$\int_{-\infty}^{0} x(t)e^{-st}dt = \lim_{T_1\to\infty} \int_{-T_1}^{0} x(t)e^{-st}dt$$

The ROC for the Laplace transform is the intersection of the ROCs for the above two one-sided integrals.

NB. The integral in ($\star$) is the unilateral Laplace transform of x.

Convergence of Unilateral Laplace Transform

Theorem. If the integral in ($\star$) converges for $s = s_0 = \sigma_0 + j\omega_0$, then it converges for any $s = \sigma + j\omega$ with $\sigma > \sigma_0$.

Proof. Let

$$y(t) = \int_0^t x(\tau)e^{-s_0\tau}d\tau$$

By assumption, $\lim_{t\to\infty} y(t)$ exists and hence $M \triangleq \sup_{t\geq 0} |y(t)| < \infty$.

Integration by parts yields

$$\begin{aligned}\int_0^T x(t)e^{-st}dt &= \int_0^T y'(t)e^{-(s-s_0)t}dt \\ &= e^{-(s-s_0)T}y(T) + (s-s_0)\int_0^T y(t)e^{-(s-s_0)t}dt\end{aligned}$$

As $T \to \infty$, $e^{-(s-s_0)T}y(T) \to 0$.

Convergence of Unilateral Laplace Transform

Theorem. If the integral in ($\star$) converges for $s = s_0 = \sigma_0 + j\omega_0$, then it converges for any s with $\operatorname{Re} s > \sigma_0$.

Proof (cont'd). Let $s = \sigma + j\omega$.

$$\begin{aligned}\int_0^T |y(t)e^{-(s-s_0)t}|dt &= \int_0^T |y(t)|e^{-(\sigma-\sigma_0)t}dt \\ &\le \int_0^T Me^{-(\sigma-\sigma_0)t}dt \le \frac{M}{\sigma-\sigma_0}\end{aligned}$$

so the integral $\int_0^\infty y(t)e^{-(s-s_0)t}dt$ converges absolutely.

Therefore, ($\star$) converges and

$$\int_0^\infty x(t)e^{-st}dt = (s-s_0)\int_0^\infty y(t)e^{-(s-s_0)t}dt$$

ROC of Unilateral Laplace Transform

Three possibilities for the convergence of the integral ($\star$),

(a). it converges for every $s \in \mathbb{C}$

(b). it diverges for every $s \in \mathbb{C}$

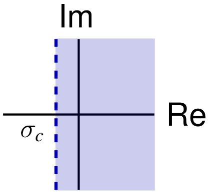
(c). it converges for $\text{Re}\, s > \sigma_c \in \mathbb{R}$ and diverges for $\text{Re}\, s < \sigma_c$

In case (c), $\sigma_c \in \mathbb{R}$ is called the abscissa of convergence, and the line $\text{Re}\, s = \sigma_c$ is called the axis of convergence

The ROC[1] is always a right half-plane

$$\text{ROC} = \{s \in \mathbb{C} : \text{Re}\, s > \sigma_c\}$$

We also write $\sigma_c = -\infty$ in case (a), and $\sigma_c = +\infty$ in case (b).



[1] More precisely, the interior of the ROC.

ROC of Unilateral Laplace Transform

Example. $\int_0^\infty e^{t^2} e^{-st} dt$ diverges for every $s \in \mathbb{C}$, i.e. $\sigma_c = +\infty$

Proof. Let $s = \sigma \in \mathbb{R}$. Note

$$\int_0^\infty e^{t^2} e^{-\sigma t} dt = +\infty$$

Since σ is arbitrary, the previous theorem implies the integral diverges for any $s \in \mathbb{C}$.

Example. $\int_0^\infty e^{-t^2} e^{-st} dt$ converges for every $s \in \mathbb{C}$, i.e. $\sigma_c = -\infty$

Proof. Let $s = \sigma \in \mathbb{R}$. Note

$$\int_0^\infty e^{-t^2} e^{-\sigma t} dt = e^{\sigma^2/4} \int_0^\infty e^{-(t+\sigma/2)^2} dt \le e^{\sigma^2/4} \int_{-\infty}^\infty e^{-t^2} dt = \sqrt{\pi} e^{\sigma^2/4}$$

The integral converges absolutely for every $s \in \mathbb{C}$.

Example. $\int_0^\infty e^{-st} dt$ has $\sigma_c = 0$

Absolute Convergence of Unilateral Laplace Transform

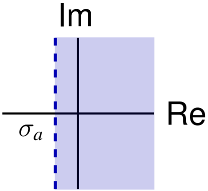
Theorem. If the integral in ($\star$) converges absolutely for $s = s_0 = \sigma_0 + j\omega_0$, then it converges absolutely and uniformly for $s = \sigma + j\omega$ with $\sigma \geq \sigma_0$.

Proof.

$$\int_0^\infty |x(t)e^{-st}|dt = \int_0^\infty |x(t)|e^{-\sigma t}dt \leq \int_0^\infty |x(t)|e^{-\sigma_0 t}dt < \infty$$

As in the case of convergence, the region of absolute convergence (ROAC) is always a right half-plane

$$\text{ROAC} = \{s \in \mathbb{C} : \operatorname{Re} s > \sigma_a\}$$



where $\sigma_a \in \bar{\mathbb{R}}$ is the abscissa of absolute convergence, and the line $\operatorname{Re} s = \sigma_a$ is the axis of absolute convergence.

ROAC of Unilateral Laplace Transform

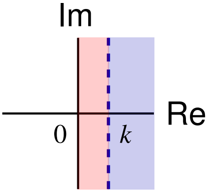
The axis of convergence and the axis of absolute convergence **need not coincide**!

Example. Let $k > 0$. To see $\sigma_a = k$, note $\int_0^\infty e^{kt}\sin(e^{kt})e^{-st}dt$ converges absolutely for Re $s > k$, since for $s = \sigma + j\omega$,

$$|e^{kt}\sin(e^{kt})e^{-st}| \le e^{-(\sigma-k)t} \in L_1[0,\infty)$$

It is not absolutely convergent for $s = k$, since

$$\int_0^\infty |\sin(e^{kt})|dt = \frac{1}{k}\int_1^\infty \frac{|\sin u|}{u}du = \infty.$$



Let $s = \sigma \in \mathbb{R}$.

$$\int_0^\infty e^{kt}\sin(e^{kt})e^{-\sigma t}dt = \frac{1}{k}\int_1^\infty \frac{\sin u}{u^{\sigma/k}}du$$

Dirichlet's test implies $\sigma_c = 0$.

NB. We will use ROC, although in most cases ROC = ROAC.

ROC of (Bilateral) Laplace Transform

The ROC of

$$\int_0^\infty x(t)e^{-st}dt$$

is a right half-plane $\operatorname{Re} s > \sigma_1$.

The ROC of

$$\int_{-\infty}^0 x(t)e^{-st}dt = \int_0^\infty x(-t)e^{-(-s)t}dt$$

is a left half-plane, $\operatorname{Re} s < \sigma_2$.

The ROC of

$$X(s) = \int_{-\infty}^\infty x(t)e^{-st}dt = \int_0^\infty x(t)e^{-st}dt + \int_{-\infty}^0 x(t)e^{-st}dt$$

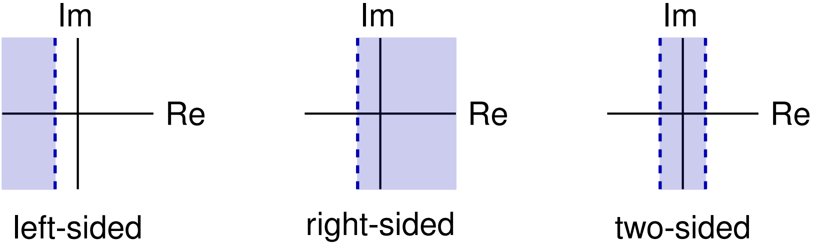
is a strip $\sigma_1 < \operatorname{Re} s < \sigma_2$, which is nonempty iff $\sigma_1 < \sigma_2$

Properties of ROC

- $X(s)$ is analytic in the ROC and

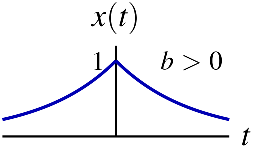
$$f^{(k)}(s) = \int (-t)^k x(t) e^{-st} dt$$

- If x is of finite duration and in L_1, then the ROC is $\mathbb{C}$.
- If x is left-sided, its ROC is a left half-plane
- If x is right-sided, its ROC is a right half-plane
- If x is two-sided, its ROC is a strip



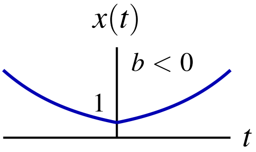
Example

Consider $x(t) = e^{-b|t|} = e^{-bt}u(t) + e^{bt}u(-t)$



Recall

$$e^{-bt}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+b}, \quad \text{Re}\, s > -b$$

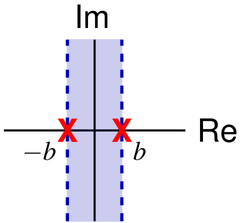


$$e^{bt}u(-t) \xleftrightarrow{\mathcal{L}} \frac{-1}{s-b}, \quad \text{Re}\, s < b$$

Thus

$$x(t) = e^{-b|t|} \xleftrightarrow{\mathcal{L}} \frac{1}{s+b} - \frac{1}{s-b} = \frac{-2b}{s^2 - b^2}, \quad -b < \text{Re}\, s < b$$

The ROC is nonempty iff $b > 0$.



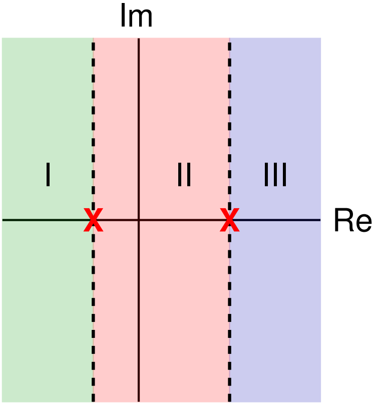
NB. For Laplace transform to exist, $x(t)$ has to decay fast enough either as $t \to +\infty$ or $t \to -\infty$. The exponential weighting in e^{-st} cannot kill both.

Properties of ROC

- If $X(s)$ is rational, the ROC is bounded by poles or extends to infinity.

- If x is also right-sided, then its ROC is the right half-plane bounded by the rightmost pole in $\mathbb{C}$, e.g. region III

- If x is also left-sided, then its ROC is the left half-plane bounded by the leftmost pole in $\mathbb{C}$, e.g. region I



Contents

Linearity

If

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \quad \text{with ROC} = R_1$$

$$y(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s) \quad \text{with ROC} = R_2$$

then

$$ax(t) + by(t) \overset{\mathcal{L}}{\longleftrightarrow} aX(s) + bY(s) \quad \text{with ROC} \supset R_1 \cap R_2$$

NB. The ROC may be larger than $R_1 \cap R_2$.

Example. $x(t) = \cos(\omega_0 t)u(t) = \frac{1}{2}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-j\omega_0 t}u(t)$

$$X(s) = \frac{1}{2}\frac{1}{s - j\omega_0} + \frac{1}{2}\frac{1}{s + j\omega_0} = \frac{s}{s^2 + \omega_0^2}, \quad \text{Re}\, s > 0$$

$$\sin(\omega_0 t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{\omega_0}{s^2 + \omega_0^2}, \quad \text{Re}\, s > 0$$

Linearity

Example.

$$X_1(s) = \frac{1}{s+1}, \quad \text{Re}\, s > -1; \quad X_2(s) = \frac{1}{(s+1)(s+2)}, \quad \text{Re}\, s > -1$$

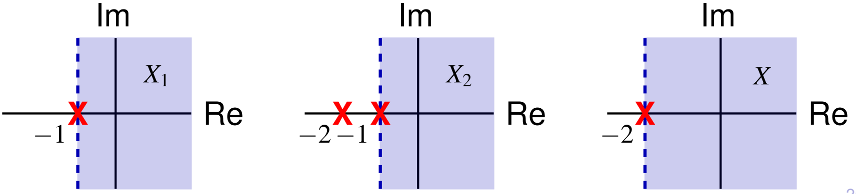
$$X(s) = X_1(s) - X_2(s) = \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2}, \quad \text{Re}\, s > -2$$

ROC enlarges due to pole-zero cancellation at $s = -1$.

In time domain,

$$x_1(t) = e^{-t}u(t), \quad x_2(t) = e^{-t}u(t) - e^{-2t}u(t)$$

$$x(t) = x_1(t) - x_2(t) = e^{-2t}u(t)$$



Time Shift

If

$$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s), \quad \sigma_1 < \text{Re}\, s < \sigma_2$$

then

$$x(t - t_0) \stackrel{\mathcal{L}}{\longleftrightarrow} e^{-st_0} X(s), \quad \sigma_1 < \text{Re}\, s < \sigma_2$$

Example.

$$? \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{e^{3s}}{s+2}, \ \text{Re}\, s > -2$$

$$x(t) = e^{-2t}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \ \text{Re}\, s > -2$$

$$x(t+3) = e^{-2(t+3)}u(t+3) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{e^{3s}}{s+2}, \ \text{Re}\, s > -2$$

Shifting in s-domain

If

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad \sigma_1 < \text{Re}\, s < \sigma_2$$

then

$$e^{s_0 t} x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s - s_0), \quad \sigma_1 + \text{Re}\, s_0 < \text{Re}\, s < \sigma_2 + \text{Re}\, s_0$$

Example.

For $\alpha \in \mathbb{R}$,

$$e^{-\alpha t} \cos(\omega_0 t) u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s + \alpha}{(s + \alpha)^2 + \omega_0^2}, \quad \text{Re}\, s > -\alpha$$

$$e^{-\alpha t} \sin(\omega_0 t) u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{\omega_0}{(s + \alpha)^2 + \omega_0^2}, \quad \text{Re}\, s > -\alpha$$

Time Scaling

If

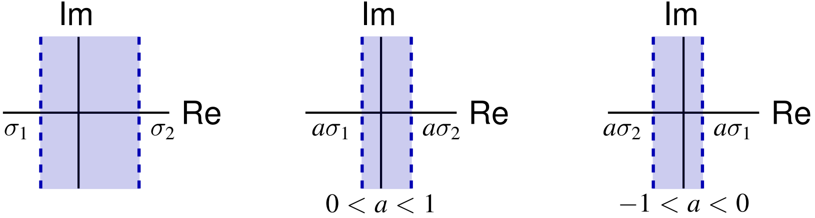
$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad \sigma_1 < \text{Re}\, s < \sigma_2$$

then for $a \in \mathbb{R} \setminus \{0\}$,

$$x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right), \quad \sigma_1 < \frac{1}{a}\text{Re}\, s < \sigma_2$$

For time reversal,

$$x(-t) \overset{\mathcal{L}}{\longleftrightarrow} X(-s), \quad -\sigma_2 < \text{Re}\, s < -\sigma_1$$



Conjugation

If

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad \sigma_1 < \text{Re}\, s < \sigma_2$$

then

$$x^*(t) \overset{\mathcal{L}}{\longleftrightarrow} X^*(s^*), \quad \sigma_1 < \text{Re}\, s < \sigma_2$$

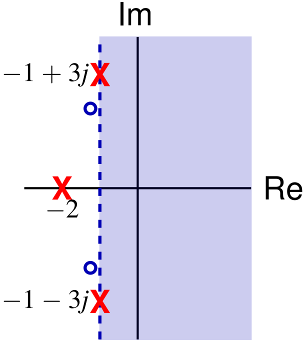
If x is real-valued, then $X(s) = X^*(s^*)$, so the zeros and poles of $X(s)$ appear in conjugate pairs.

Example.

For $x(t) = e^{-2t}u(t) + e^{-t}\cos(3t)u(t)$,

$$X(s) = \frac{(s + \frac{5+j\sqrt{71}}{4})(s + \frac{5-j\sqrt{71}}{4})}{(s+2)(s+1-3j)(s+1+3j)}$$

with ROC $\text{Re}\, s > -1$.



Convolution Property

If

$$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s) \quad \text{with ROAC} = R_1$$

$$y(t) \stackrel{\mathcal{L}}{\longleftrightarrow} Y(s) \quad \text{with ROAC} = R_2$$

then

$$(x * y)(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s)Y(s) \quad \text{with ROAC} \supset R_1 \cap R_2$$

A more precise statement is the following.

Theorem. If both $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$ and $Y(s) = \int_{-\infty}^{\infty} y(t)e^{-st}dt$ converges absolutely at some $s = s_0$, then the Laplace transform of $z = x * y$ converges absolutely at $s = s_0$, and

$$X(s_0)Y(s_0) = Z(s_0) = \int_{-\infty}^{\infty} z(t)e^{-s_0 t}dt$$

Convolution Property

Proof.

$$\begin{aligned}X(s)Y(s) &= \int_{-\infty}^{\infty} x(v)\left[\int_{-\infty}^{\infty} y(\tau)e^{-s(v+\tau)}dv\right]d\tau \\ &= \int_{-\infty}^{\infty} x(v)\left[\int_{-\infty}^{\infty} y(t-v)e^{-sv}dv\right]dt \quad (t = v+\tau) \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(v)y(t-v)dt\right]e^{-sv}dv \quad \text{(Fubini's Theorem)}\end{aligned}$$

NB. The ROAC of $\mathcal{L}\{x * y\}$ may be larger than the common ROAC of $\mathcal{L}\{x\}$ and $\mathcal{L}\{y\}$.

Example. $X_1(s) = \frac{s+1}{(s+2)^2}$ has ROAC $\operatorname{Re} s > -2$, $X_2(s) = \frac{1}{s+1}$ has ROAC $\operatorname{Re} s > -1$, but $X(s) = X_1(s)X_2(s) = \frac{1}{(s+2)^2}$ with ROAC $\operatorname{Re} s > -2$, due to pole-zero cancellation at $s = -1$.

Differentiation in Time Domain

If

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad \text{with ROC} = R$$

and $\lim_{t\to\pm\infty} x(t)e^{-st} = 0$ for $s \in R \in R_0$, then

$$\frac{d}{dt}x(t) \overset{\mathcal{L}}{\longleftrightarrow} sX(s), \quad \text{with ROC} \supset R \cap R_0$$

Proof. Integration by parts yields

$$\int_{-\infty}^{\infty} x'(t)e^{-st}dt = x(t)e^{-st}\Big|_{-\infty}^{\infty} + s\int_{-\infty}^{\infty} x(t)e^{-st}dt = s\int_{-\infty}^{\infty} x(t)e^{-st}dt$$

NB. ROC may enlarge or **shrink**

Example. $x(t) = (1-e^{-t})u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s(s+1)}$ with ROC = ROAC Re $s > 0$, and $x'(t) = e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}$ with ROC = ROAC Re $s > -1$. The ROC of $\mathcal{L}\{x'\}$ is larger that that of $\mathcal{L}\{x\}$.

Differentiation in Time Domain

Example. Consider $x(t) = e^{kt}\sin(e^{kt})$ with $k > 0$.

- For $s = \sigma \in \mathbb{R}$, $u = e^{kt}$ yields (cf. slide 18)

$$\int_{-\infty}^{\infty} x(t)e^{-st}dt = \frac{1}{k}\int_0^{\infty} \frac{\sin u}{u^{\sigma/k}}du = \frac{1}{k}\int_0^{1} \frac{\sin u}{u^{\sigma/k}}du + \frac{1}{k}\int_1^{\infty} \frac{\sin u}{u^{\sigma/k}}du$$

  - $\int_1^\infty \frac{\sin u}{u^{\sigma/k}}du$ has ROAC Re $s > k$ and ROC Re $s > 0$
  - As $u \downarrow 0$, $\frac{\sin u}{u^{\sigma/k}} \sim u^{1-\sigma/k}$, so $\int_0^1 \frac{\sin u}{u^{\sigma/k}}du$ has ROAC Re $s < 2k$
  - Thus $\mathcal{L}\{x\}$ has ROC $k <$ Re $s < 2k$ and ROC $0 <$ Re $s < 2k$.

- $x'(t) = ke^{kt}\sin(e^{kt}) + ke^{2kt}\cos(e^{kt})$. For $s = \sigma \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} x'(t)e^{-st}dt = \int_0^{\infty} \frac{\sin u + u\cos u}{u^{\sigma/k}}du$$

  $\mathcal{L}\{x'\}$ has empty ROAC, and ROC $k <$ Re $s < 2k$

- Note $\lim\limits_{t\to\pm\infty} x(t)e^{-st} = 0$ fails for s with $0 <$ Re $s < k$

Differentiation in Time Domain

If $x(t) = O(e^{at})$ as $t \to +\infty$ and $x(t) = O(e^{bt})$ as $t \to -\infty$, then

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad \text{with ROAC containing } a < \operatorname{Re} s < b$$

and

$$\frac{d}{dt}x(t) \overset{\mathcal{L}}{\longleftrightarrow} sX(s), \quad \text{with ROC containing } a < \operatorname{Re} s < b$$

NB. In general, from the absolute convergence of $\mathcal{L}\{x\}$ at $s = s_0$ we can only conclude the convergence of $\mathcal{L}\{x'\}$ at $s = s_0$.

NB. We mostly deal with $x(t)$ of the form $\sum_{k=0}^{m} p_k(t) e^{\alpha_k t} u(\pm t + \beta_k)$, where p_k are polynomials. After introducing Laplace transform for singularity functions, we have for such functions,

$$\frac{d^n}{dt^n}x(t) \overset{\mathcal{L}}{\longleftrightarrow} s^n X(s), \quad \text{with ROC containing } a < \operatorname{Re} s < b$$

Differentiation in s-domain

If
$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad \sigma_1 < \text{Re}\, s < \sigma_2$$
then
$$-tx(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{d}{ds}X(s), \quad \sigma_1 < \text{Re}\, s < \sigma_2$$

Proof. Differentiation under integral sign (can be justified) yields

$$\frac{d}{ds}\int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} x(t)\frac{d}{ds}e^{-st}dt = \int_{-\infty}^{\infty} -tx(t)e^{-st}dt$$

Example.
$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \text{Re}\, s > -a$$
$$t^n e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \left(-\frac{d}{ds}\right)^n \frac{1}{s+a} = \frac{n!}{(s+a)^{n+1}}, \quad \text{Re}\, s > -a$$

Similarly,
$$-t^n e^{-at}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{n!}{(s+a)^{n+1}}, \quad \text{Re}\, s < -a$$