EI331 Signals and Systems

Lecture 3

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Contents

Complex Function

Complex function

$$\begin{aligned} f : G \subset \mathbb{C} &\to \mathbb{C} \\ z &\mapsto f(z) \end{aligned}$$

Four kinds of complex functions

1. real function of a real variable
   - studied in calculus. e.g. $f(t) = e^t, t \in \mathbb{R}$
2. complex function of a real variable
   - current focus. e.g. $f(t) = e^{j2\pi t}, t \in \mathbb{R}$
3. real function of a complex variable
   - e.g. $f(z) = |z|, z \in \mathbb{C}$
4. complex function of a complex variable
   - later, e.g. $f(z) = e^z, z \in \mathbb{C}$

Complex Function of a Real Variable

$$\begin{aligned} f : G \subset \mathbb{R} &\to \mathbb{C} \\ t &\mapsto f(t) \end{aligned}$$

Equivalent to two real functions of a real variable

$$f(t) = u(t) + jv(t) \iff \begin{cases} u(t) = \text{Re}\, f(t) \\ v(t) = \text{Im}\, f(t) \end{cases}$$

e.g.

$$f(t) = e^{j2\pi t} \iff \begin{cases} u(t) = \cos(2\pi t) \\ v(t) = \sin(2\pi t) \end{cases}$$

Complex Function of a Real Variable

Calculus of $f(t) = u(t) + jv(t)$

Limit

$$\lim_{t \to t_0} f(t) = \lim_{t \to t_0} u(t) + j \lim_{t \to t_0} v(t)$$

Continuity

$$f(t) \text{ continuous} \iff u(t) \text{ and } v(t) \text{ continuous}$$

Differentiation

$$f'(t) = u'(t) + jv'(t)$$

Integration

$$\int f(t)dt = \int u(t)dt + j \int v(t)dt$$

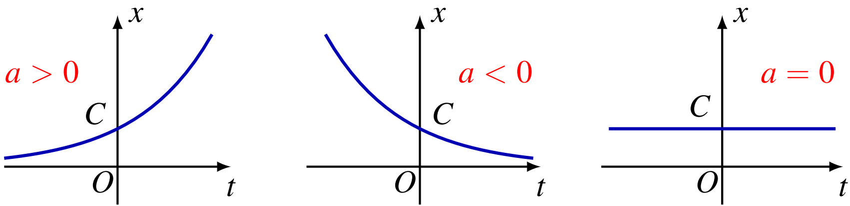
Contents

CT Complex Exponential Signals

$$x(t) = Ce^{at}, \quad \text{where } C \in \mathbb{C}, a \in \mathbb{C}$$

Real exponential signals: $C \in \mathbb{R}$, $a \in \mathbb{R}$

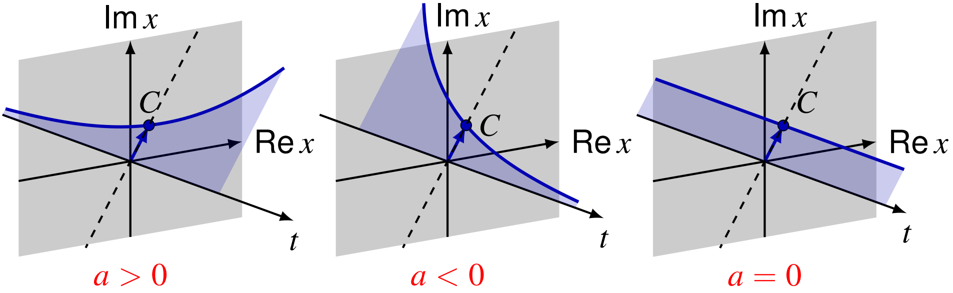
1. $a > 0$: growing exponential
2. $a < 0$: decaying exponential
3. $a = 0$: constant



CT Complex Exponential Signals

$$x(t) = Ce^{at}, \quad \text{where } C \in \mathbb{C}, a \in \mathbb{R}$$

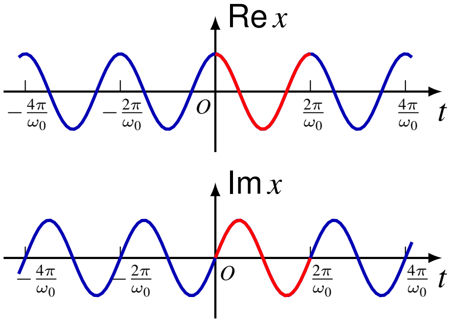
1. $a > 0$: diverges from t axis, $|x(t)| \nearrow \infty$ as $t \to \infty$
2. $a < 0$: converges to t axis, $|x(t)| \searrow 0$ as $t \to \infty$
3. $a = 0$: constant

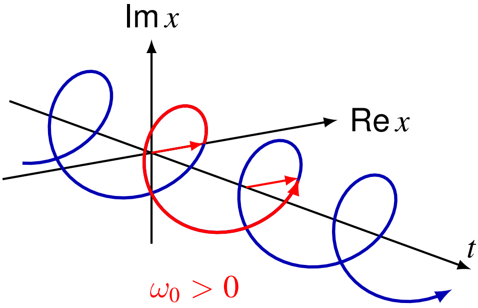


CT Complex Exponential Signals

Periodic complex exponential signals

$$x(t) = e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t), \quad \text{where } \omega_0 \in \mathbb{R}$$





(Angular) frequency: ω_0 radians/s

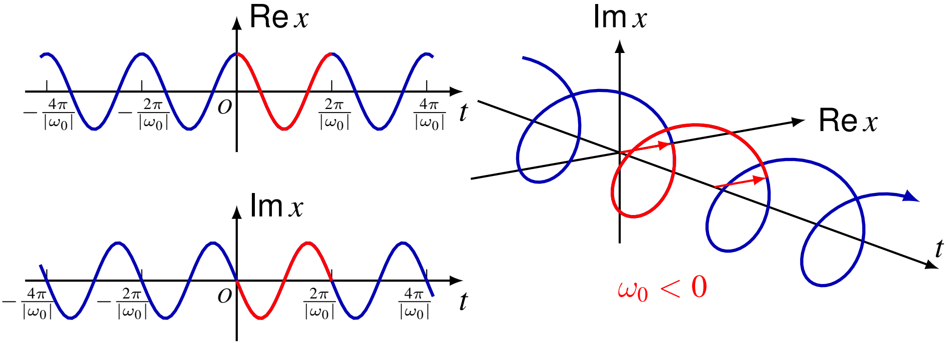
Frequency: $f_0 = \omega_0/(2\pi)$ cycles/s, Hz

Fundamental period: $T_0 = 2\pi/|\omega_0| = 1/|f_0|$ s **(only if $\omega_0 \neq 0$)**

CT Complex Exponential Signals

Periodic complex exponential signals

$$x(t) = e^{j\omega_0 t} = \cos(|\omega_0|t) - j\sin(|\omega_0|t), \quad \text{where } \omega_0 < 0$$

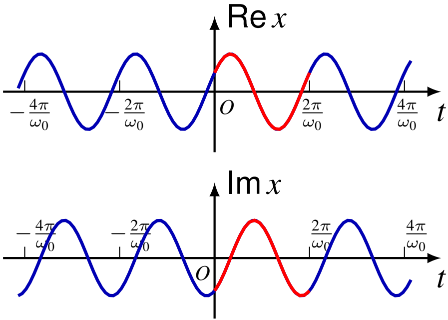


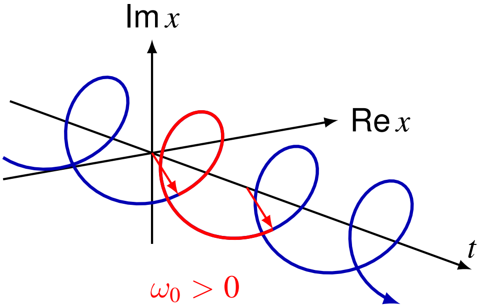
Fundamental frequency: $|\omega_0|$, $|f_0|$

CT Complex Exponential Signals

Periodic complex exponential signals

$$x(t) = Ce^{j\omega_0 t} = |C|\cos(\omega_0 t+\phi)+j\sin(\omega_0 t+\phi), \quad \text{where } C = |C|e^{j\phi}$$





CT Complex Exponential Signals

Sinusoidal signals

$$x(t) = A\cos(\omega_0 t + \phi), \quad A \in \mathbb{R}$$

Conversion between exponentials and sinusoids

$$Ae^{j(\omega_0 t + \phi)} = A\cos(\omega_0 t + \phi) + jA\sin(\omega_0 t + \phi)$$

$$A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 t} = A \cdot \text{Re}\, e^{j(\omega_0 t + \phi)}$$

$$A\sin(\omega_0 t + \phi) = A \cdot \text{Im}\, e^{j(\omega_0 t + \phi)}$$

Same periodicity

- always periodic with fundamental frequency $|\omega_0|$
- larger $|\omega_0|$, faster oscillation

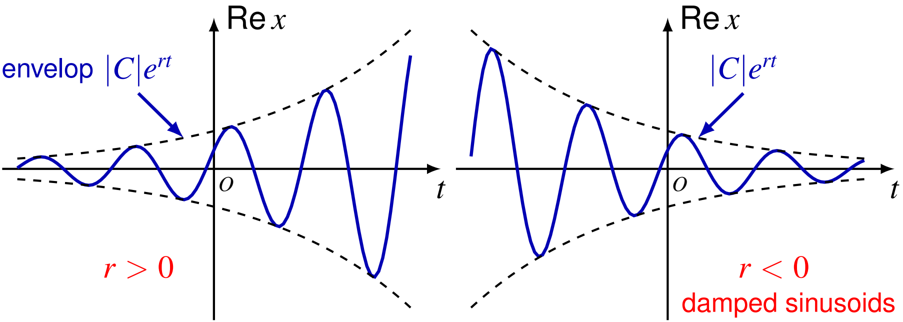
CT Complex Exponential Signals

General Complex Exponential Signals

$$x(t) = Ce^{at}, \quad \text{where } C = |C|e^{j\phi}, a = r + j\omega_0$$

$$\Downarrow$$

$$x(t) = |C|e^{rt}e^{j(\omega_0 t+\phi)} = |C|e^{rt}\cos(\omega_0 t + \phi) + j|C|e^{rt}\sin(\omega_0 t + \phi)$$

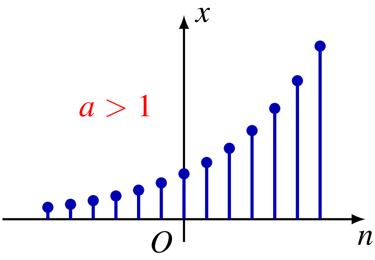


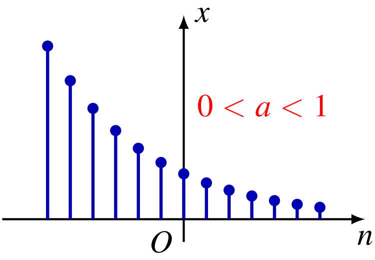
DT Complex Exponential Signals

$$x[n] = C\alpha^n = Ce^{\beta n}, \quad \text{where } C \in \mathbb{C}, \alpha = e^{\beta} \in \mathbb{C}$$

Real exponential signals: $C \in \mathbb{R}$, $\alpha \in \mathbb{R}$ **(but $\beta \in \mathbb{C}$!)**

1. $\alpha > 1$: monotonically growing
2. $0 < \alpha < 1$: monotonically decaying
3. $\alpha = 1$: constant



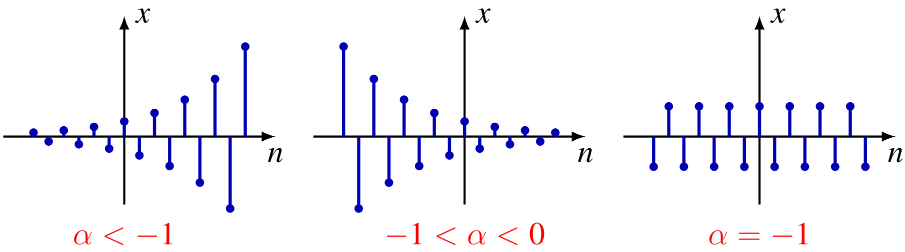


DT Complex Exponential Signals

$$x[n] = C\alpha^n = Ce^{\beta n}, \quad \text{where } C \in \mathbb{C}, \alpha = e^{\beta} \in \mathbb{C}$$

Real exponential signals: $C \in \mathbb{R}$, $\alpha \in \mathbb{R}$ **(but $\beta \in \mathbb{C}$!)**

4. $\alpha < -1$: growing magnitude, alternating sign
5. $-1 < \alpha < 0$: decaying magnitude, alternating sign
6. $\alpha = -1$: constant magnitude, alternating sign **($\beta = j\pi$)**



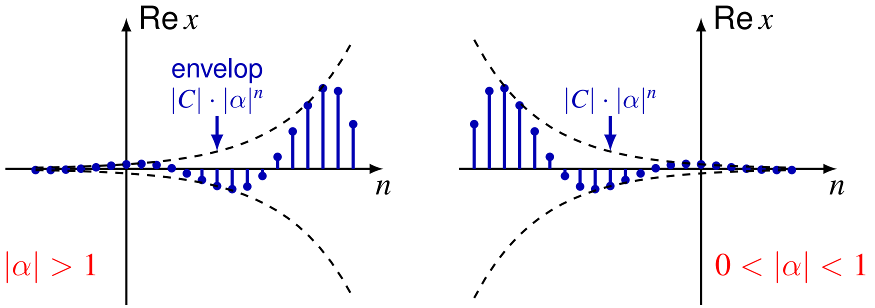
DT Complex Exponential Signals

General Complex Exponential Signals

$$x[n] = C\alpha^n, \quad \text{where } C = |C|e^{j\phi}, \alpha = |\alpha|e^{j\omega_0}$$

$$\Downarrow$$

$$x[n] = |C|\cdot|\alpha|^n e^{j(\omega_0 n+\phi)} = |C|\cdot|\alpha|^n \cos(\omega_0 n+\phi)+j|C|\cdot|\alpha|^n \sin(\omega_0 n+\phi)$$



DT Complex Exponential Signals

Sinusoidal Signals

$$x[n] = |C|e^{j(\omega_0 n+\phi)} = |C|\cos(\omega_0 n + \phi) + j|C|\sin(\omega_0 n + \phi)$$

Periodicity

- periodic $\iff \omega_0 = 2\pi\frac{k}{N}$ for $k \in \mathbb{Z}, N \in \mathbb{Z}_+$
- fundamental period $N_0 = N/\gcd(N, k)$

Fundamental frequency

- zero if $N_0 = 1$
- $2\pi/N_0$ if $N_0 > 1$

Example. $x[n] = e^{j3\pi n}$ has $N_0 = 2$, fundamental frequency π, **not** 3π**!** Note $e^{j3\pi n} = e^{j\pi n}$.

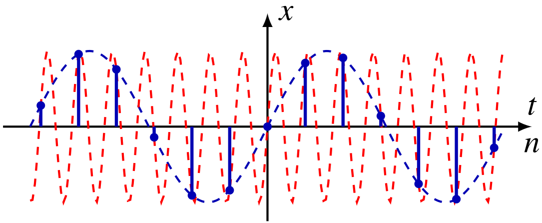
DT Complex Exponential Signals

Aliasing

- $e^{j\omega_1 t} = e^{j\omega_2 t}, \forall t \in \mathbb{R} \iff \omega_1 = \omega_2$
- $e^{j\omega_1 n} = e^{j\omega_2 n}, \forall n \in \mathbb{N} \iff \omega_1 = \omega_2 + 2k\pi, k \in \mathbb{Z}$

frequencies differing by $2k\pi$ yields same discrete sinusoid

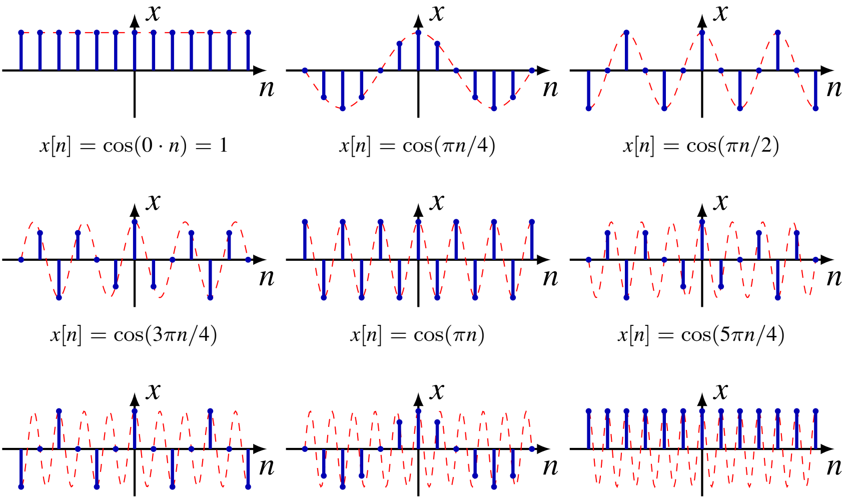
Example. $\omega_1 = 1$, $\omega_2 = 1 + 2\pi$



For DT signals, suffices to consider frequencies on an interval of length 2π, e.g. $[0, 2\pi)$ or $(-\pi, \pi]$

DT Complex Exponential Signals

High frequencies around $(2k+1)\pi$, low frequencies around $2k\pi$



$x[n] = \cos(0 \cdot n) = 1$ $\quad$ $x[n] = \cos(\pi n/4)$ $\quad$ $x[n] = \cos(\pi n/2)$

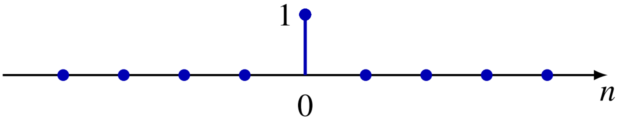
$x[n] = \cos(3\pi n/4)$ $\quad$ $x[n] = \cos(\pi n)$ $\quad$ $x[n] = \cos(5\pi n/4)$

$x[n] = \cos(3\pi n/2)$ $\quad$ $x[n] = \cos(7\pi n/4)$ $\quad$ $x[n] = \cos(2\pi n) = 1$

Contents

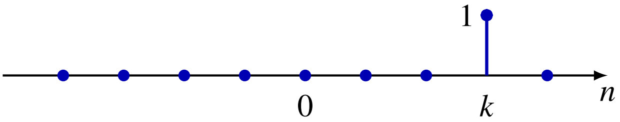
DT Unit Impulse (Unit Sample)

$$\delta[n] = \delta_{0n} = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



Shifted version $\tau_k \delta$

$$\tau_k \delta[n] = \delta[n-k] = \delta_{kn} = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases}$$



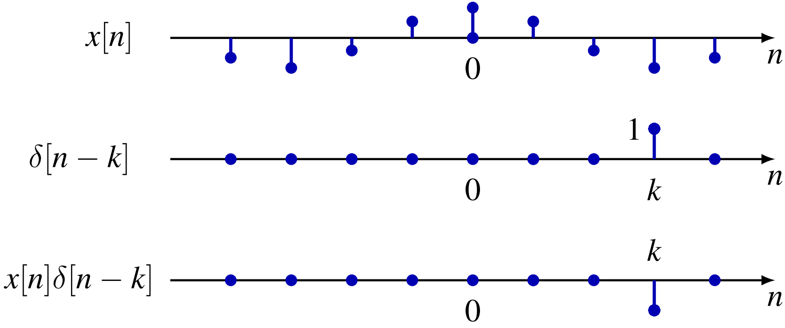
DT Unit Impulse (Unit Sample)

Sampling property

$$x\delta = x[0]\delta, \quad \text{or} \quad (x\delta)[n] = x[n]\delta[n] = x[0]\delta[n], \ \forall n \in \mathbb{Z}$$

More generally,

$$x\tau_k\delta = x[k]\tau_k\delta, \quad \text{or} \quad x[n]\delta[n-k] = x[k]\delta[n-k], \ \forall n \in \mathbb{Z}$$



DT Unit Impulse (Unit Sample)

Recall from linear algebra,

$$\mathbf{e}_k = (0, \ldots, 0, \underset{\substack{\uparrow \\ k\text{-th}}}{1}, 0, \ldots, 0)^T, \quad k = 1, 2, \ldots, n$$

form basis of $\mathbb{R}^n$ (and $\mathbb{C}^n$)

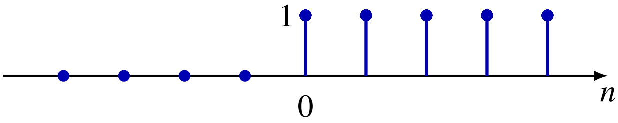
$$\mathbf{x} = (x_1, x_2, \ldots, x_n)^n = \sum_{k=1}^{n} x_k \mathbf{e}_k$$

Similarly, $\{\tau_k \delta : k \in \mathbb{Z}\}$ is a basis of $\mathbb{C}^{\mathbb{Z}}$, space of doubly infinite sequences of complex numbers

$$x = \sum_{k=-\infty}^{\infty} x[k] \tau_k \delta, \quad \textbf{or} \quad x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k], \ \forall n \in \mathbb{Z}$$

DT Unit Step

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



Relation to unit impulse

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] = \sum_{m=-\infty}^{n} \delta[m] \quad \text{running sum}$$

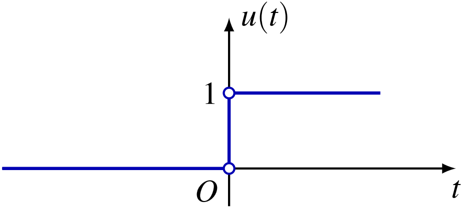
$$\delta[n] = u[n] - u[n-1] \quad \text{first (backward) difference}$$

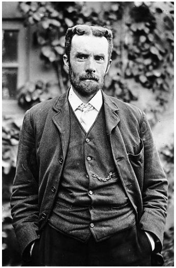
CT Unit Step Function

Also called Heaviside (step) function

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

- undefined at $t = 0$
- sometimes $u(0) = 0, 1, 1/2$





Oliver Heaviside
(from Wikipedia)

CT Unit Impulse Function

Also called Dirac delta function or δ function

$$\delta(t) = \frac{du}{dt}(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

where

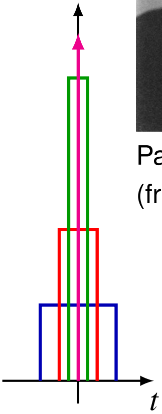
$$\delta_\Delta(t) = \frac{u(t + \frac{\Delta}{2}) - u(t - \frac{\Delta}{2})}{\Delta}$$

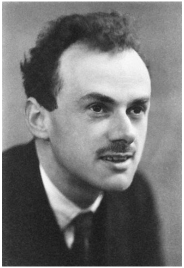
Physical models

- density of point mass/charge
- impulse force

By calculus

$$\frac{du}{dt}(t) = \begin{cases} 0, & t \neq 0 \\ +\infty, & t = 0 \end{cases}$$





Paul Dirac
(from Wikipedia)

Specification of Function by Action

Classically, function is defined by specifying value at each point in its domain, e.g.

$$\begin{aligned} x :\ & \mathbb{R} \to \mathbb{R} \\ & t \mapsto x(t) \end{aligned}$$

Idea. Define function by “action” on “test functions”

Example in physics

- impulse force specified by change of momentum

Example in linear algebra

- Matrix $A \in \mathbb{R}^{n\times n}$
  - specify each entry A_{ij}
  - specify action $A\mathbf{x}$ for all vector $\mathbf{x} \in \mathbb{R}^n$

Specification of Function by Action

Let $C[a,b]$ denote set of continuous functions on $[a,b]$

Action of x on $\phi \in C[a,b]$

$$T_x[\phi] = \int_a^b x(t)\phi(t)dt \quad \text{functional on } C[a,b]$$

Lemma. Let $x_i(t) \in C[a,b]$, $i = 1, 2$.

$$T_{x_1}[\phi] = T_{x_2}[\phi],\ \forall \phi \in C[a,b] \iff x_1 = x_2$$

Proof. $\Leftarrow$ obvious. For $\Rightarrow$, let $\phi = \bar{x}_1 - \bar{x}_2$. Then

$$\int_a^b |x_1(t) - x_2(t)|^2 dt = 0 \implies |x_1(t) - x_2(t)|^2 = 0 \implies x_1 = x_2$$

Set of values $T_x[\phi], \forall \phi \in C[a,b]$ uniquely determines x!

To determine x, instead of specifying $x(t)$ for all $t \in [a,b]$, specify $T_x[\phi]$ for all $\phi \in C[a,b]$.

Action of Unit Impulse Function

Define action of δ on ϕ by

$$\delta[\phi] = \text{“}\int_{-\infty}^{\infty} \delta(t)\phi(t)dt\text{”} \triangleq \lim_{\Delta\to\infty} T_{\delta_\Delta}[\phi]$$

for $\phi(t)$ **continuous at $t = 0$**.

1. Compute

$$T_{\delta_\Delta}[\phi] = \int_{-\infty}^{\infty} \delta_\Delta(t)\phi(t)dt = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} \phi(t)dt$$

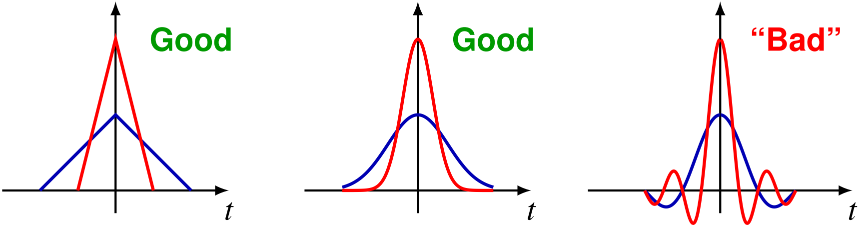
2. Let $\Delta \to 0$

$$\phi \text{ continuous at } 0 \implies \lim_{\Delta\to 0} T_{\delta_\Delta}[\phi] = \phi(0)$$

Defining property of δ: $\delta[\phi] \triangleq \int_{-\infty}^{\infty} \delta(t)\phi(t)dt = \phi(0)$

Other Approximations

Can define δ as limit of other functions.



$$g_\Delta(t) = \frac{1}{\sqrt{2\pi}\Delta} e^{-\frac{t^2}{2\Delta^2}} \qquad D_\Delta(t) = \frac{\sin\left(\frac{\pi t}{\Delta}\right)}{\pi t}$$

Family $\{K_\Delta(t)\}_{\Delta>0}$ called good kernels or approximation to the identity if

1. For all $\Delta > 0$, $\int_{-\infty}^{\infty} K_\Delta(t)dt = 1$
2. For some $M > 0$ and all $\Delta > 0$, $\int_{-\infty}^{\infty} |K_\Delta(t)|dt < M$
3. For every $\epsilon > 0$, $\lim_{\Delta \to 0} \int_{|t|>\epsilon} |K_\Delta(t)|dx = 0$

Properties of Unit Impulse Function

Unit "area"

$$\int_{-\infty}^{\infty} \delta(\tau) d\tau = 1$$

Proof. Let $\phi(t) = 1$ in defining property.

Integration

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

Proof. Let $\phi_t(\tau) = u(t - \tau)$. For $t \neq 0$, ϕ_t is continuous at $\tau = 0$. By defining property,

$$\int_{-\infty}^{t} \delta(\tau) d\tau = \int_{-\infty}^{\infty} \delta(\tau)\phi_t(\tau) d\tau = \phi_t(0) = u(t)$$

Transformations of Unit Impulse

Defined s.t. usual rules for change of variables hold

Time scaling $(S_a\delta)[\phi] \triangleq \delta[a^{-1}S_{a^{-1}}\phi]$, where $a > 0$

$$\int_{-\infty}^{\infty} \delta(at)\phi(t)dt \triangleq \int_{-\infty}^{\infty} \delta(t)\phi\left(\frac{t}{a}\right)\frac{dt}{a} \implies \delta(at) = \frac{1}{a}\delta(t)$$

Time reversal $(R\delta)[\phi] \triangleq \delta[R\phi]$

$$\int_{-\infty}^{\infty} \delta(-t)\phi(t)dt \triangleq \int_{-\infty}^{\infty} \delta(t)\phi(-t)dt \implies \delta(-t) = \delta(t)$$

Time shift $(\tau_a\delta)[\phi] \triangleq \delta[\tau_{-a}\phi]$

$$\int_{-\infty}^{\infty} \delta(t-a)\phi(t)dt \triangleq \int_{-\infty}^{\infty} \delta(t)\phi(t+a)dt = \phi(a)$$

Multiplication and Sampling Property

Multiplication by ordinary function $(x\delta)[\phi] = \delta[x\phi]$

$$\int_{-\infty}^{\infty} [x(t)\delta(t)]\phi(t)dt \triangleq \int_{-\infty}^{\infty} \delta(t)[x(t)\phi(t)]dt = x(0)\phi(0)$$

Sampling property

$$x\delta = x(0)\delta, \quad \text{or} \quad x(t)\delta(t) = x(0)\delta(t)$$

Proof.

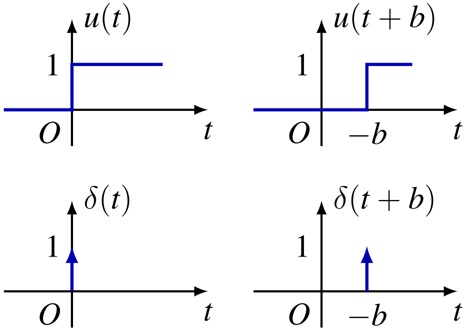
$$(x\delta)[\phi] = \delta[x\phi] = x(0)\phi(0) = x(0)\delta[\phi] = (x(0)\delta)[\phi]$$

Derivatives of $u(at+b)$ and $x(t)u(t)$

Chain rule holds

$$\frac{d}{dt}u(t+b) = \delta(t+b)$$

$$\frac{d}{dt}u(at+b) = a\delta(at+b)$$



Leibniz rule holds

$$[x(t)u(t)]' = x'(t)u(t) + x(0)\delta(t)$$

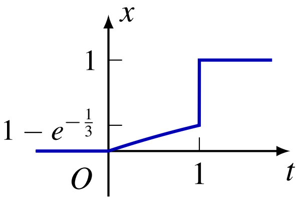
Will see later general procedure for taking derivatives.

Functions with Jump Discontinuities

Example.

$$x(t) = (1 - e^{-\frac{1}{3}t})[u(t) - u(t-1)] + u(t-1)$$

$$= \begin{cases} 0, & t < 0 \\ 1 - e^{-t/3}, & 0 < t < 1 \\ 1, & t > 1 \end{cases}$$



$$x'(t) = \frac{1}{3}e^{-\frac{1}{3}t}[u(t) - u(t-1)] + e^{-\frac{1}{3}}\delta(t-1)$$

1. impulse at each discontinuity
2. impulse size equal to jump size