EI331 Signals and Systems

Lecture 30

Bo Jiang

John Hopcroft Center for Computer Science
Shanghai Jiao Tong University

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Contents

1. Properties of Laplace Transforms

2. Inverse Laplace Transform

3. Laplace Transform of Singularity Functions

4. Analysis of CT LTI Systems by Laplace Transform

5. Block Diagram Representations

Convolution Property

If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \quad \text{with ROAC} = R_1$$

$$y(t) \xleftrightarrow{\mathcal{L}} Y(s) \quad \text{with ROAC} = R_2$$

then

$$(x * y)(t) \xleftrightarrow{\mathcal{L}} X(s)Y(s) \quad \text{with ROAC} \supset R_1 \cap R_2$$

A more precise statement is the following.

Theorem. If both $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$ and $Y(s) = \int_{-\infty}^{\infty} y(t)e^{-st}dt$ converges absolutely at some $s = s_0$, then the Laplace transform of $z = x * y$ converges absolutely at $s = s_0$, and

$$X(s_0)Y(s_0) = Z(s_0) = \int_{-\infty}^{\infty} z(t)e^{-s_0 t}dt$$

Convolution Property

Proof.

$$\begin{aligned} X(s)Y(s) &= \int_{-\infty}^{\infty} x(v)\left[\int_{-\infty}^{\infty} y(\tau)e^{-s(v+\tau)}dv\right]d\tau \\ &= \int_{-\infty}^{\infty} x(v)\left[\int_{-\infty}^{\infty} y(t-v)e^{-sv}dv\right]dt \quad (t=v+\tau) \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(v)y(t-v)dt\right]e^{-sv}dv \quad \text{(Fubini's Theorem)} \end{aligned}$$

NB. The ROAC of $\mathcal{L}\{x*y\}$ may be larger than the common ROAC of $\mathcal{L}\{x\}$ and $\mathcal{L}\{y\}$.

Example. $X_1(s) = \frac{s+1}{(s+2)^2}$ has ROAC $\operatorname{Re} s > -2$, $X_2(s) = \frac{1}{s+1}$ has ROAC $\operatorname{Re} s > -1$, but $X(s) = X_1(s)X_2(s) = \frac{1}{(s+2)^2}$ with ROAC $\operatorname{Re} s > -2$, due to pole-zero cancellation at $s = -1$.

Differentiation in Time Domain

If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad \text{with ROC} = R$$

and $\lim_{t\to\pm\infty} x(t)e^{-st} = 0$ for $s \in R_0$, then

$$\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{L}} sX(s), \quad \text{with ROC} \supset R \cap R_0$$

Proof. Integration by parts yields

$$\int_{-\infty}^{\infty} x'(t)e^{-st}dt = x(t)e^{-st}\Big|_{-\infty}^{\infty} + s\int_{-\infty}^{\infty} x(t)e^{-st}dt = s\int_{-\infty}^{\infty} x(t)e^{-st}dt$$

NB. ROC may enlarge or **shrink**

Example. $x(t) = (1 - e^{-t})u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s(s+1)}$ with ROC = ROAC $\text{Re}\, s > 0$, and $x'(t) = e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1}$ with ROC = ROAC $\text{Re}\, s > -1$. The ROC of $\mathcal{L}\{x'\}$ is larger that that of $\mathcal{L}\{x\}$.

Differentiation in Time Domain

Example. Consider $x(t) = e^{kt}\sin(e^{kt})$ with $k > 0$.

- For $s = \sigma \in \mathbb{R}$, $u = e^{kt}$ yields (cf. slide 18)

$$\int_{-\infty}^{\infty} x(t)e^{-st}dt = \frac{1}{k}\int_0^{\infty} \frac{\sin u}{u^{\sigma/k}}du = \frac{1}{k}\int_0^{1} \frac{\sin u}{u^{\sigma/k}}du + \frac{1}{k}\int_1^{\infty} \frac{\sin u}{u^{\sigma/k}}du$$

  - $\int_1^{\infty} \frac{\sin u}{u^{\sigma/k}}du$ has ROAC Re $s > k$ and ROC Re $s > 0$
  - As $u \downarrow 0$, $\frac{\sin u}{u^{\sigma/k}} \sim u^{1-\sigma/k}$, so $\int_0^1 \frac{\sin u}{u^{\sigma/k}}du$ has ROAC Re $s < 2k$
  - Thus $\mathcal{L}\{x\}$ has ROC $k <$ Re $s < 2k$ and ROC $0 <$ Re $s < 2k$.

- $x'(t) = ke^{kt}\sin(e^{kt}) + ke^{2kt}\cos(e^{kt})$. For $s = \sigma \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} x'(t)e^{-st}dt = \int_0^{\infty} \frac{\sin u + u\cos u}{u^{\sigma/k}}du$$

  $\mathcal{L}\{x'\}$ has empty ROAC, and ROC $k <$ Re $s < 2k$

- Note $\lim_{t\to\pm\infty} x(t)e^{-st} = 0$ fails for s with $0 <$ Re $s < k$

Differentiation in Time Domain

If $x(t) = O(e^{at})$ as $t \to +\infty$ and $x(t) = O(e^{bt})$ as $t \to -\infty$, then

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad \text{with ROAC containing } a < \operatorname{Re} s < b$$

and

$$\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{L}} sX(s), \quad \text{with ROC containing } a < \operatorname{Re} s < b$$

NB. In general, from the absolute convergence of $\mathcal{L}\{x\}$ at $s = s_0$ we can only conclude the convergence of $\mathcal{L}\{x'\}$ at $s = s_0$.

NB. We mostly deal with $x(t)$ of the form $\sum_{k=0}^{m} p_k(t)e^{\alpha_k t}u(\pm t + \beta_k)$, where p_k are polynomials. After introducing Laplace transform for singularity functions, we have for such functions,

$$\frac{d^n}{dt^n}x(t) \xleftrightarrow{\mathcal{L}} s^n X(s), \quad \text{with ROC containing } a < \operatorname{Re} s < b$$

Differentiation in s-domain

If
$$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s), \quad \text{ROAC } \sigma_1 < \operatorname{Re} s < \sigma_2$$
then
$$-tx(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{d}{ds}X(s), \quad \text{ROAC } \sigma_1 < \operatorname{Re} s < \sigma_2$$

"Proof". Differentiating under integral sign (can be justified)

$$\frac{d}{ds}\int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} x(t)\frac{d}{ds}e^{-st}dt = \int_{-\infty}^{\infty} -tx(t)e^{-st}dt$$

Example.
$$e^{-at}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \operatorname{Re} s > -a$$
$$t^n e^{-at}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \left(-\frac{d}{ds}\right)^n \frac{1}{s+a} = \frac{n!}{(s+a)^{n+1}}, \quad \operatorname{Re} s > -a$$

Similarly,
$$-t^n e^{-at}u(-t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{n!}{(s+a)^{n+1}}, \quad \operatorname{Re} s < -a$$

Integration in Time Domain

If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad \text{ROAC} = R$$

then

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s}X(s), \quad \text{ROAC} \supset R \cap \{\text{Re}\, s > 0\}$$

Proof. Follows from convolution property and the following

$$\int_{-\infty}^{t} x(\tau)d\tau = (x * u)(t)$$

and

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \quad \text{with ROAC} = \{\text{Re}\, s > 0\}$$

NB. This property holds also for Laplace transforms of singularity functions and will be useful computing some Laplace transforms.

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1. Properties of Laplace Transforms

2. Inverse Laplace Transform

3. Laplace Transform of Singularity Functions

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Inverse Laplace Transform

Recall Laplace transform is related to CTFT by

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} \{x(t)e^{-\sigma t}\} e^{-j\omega t} dt = \mathcal{F}\{x(t)e^{-\sigma t}\}$$

Take the inverse Fourier transform for $s = \sigma + j\omega \in$ROC,

$$x(t)e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} d\omega$$

so

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{(\sigma + j\omega)t} d\omega$$

or

$$x(t) = \frac{1}{j2\pi} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds = \lim_{A \to \infty} \frac{1}{j2\pi} \int_{\sigma - jA}^{\sigma + jA} X(s) e^{st} ds$$

Inverse Transform by Partial Fraction Expansion

For rational Laplace transform, the inverse transform can be found by partial fraction expansion.

Recall a proper rational function has the following partial fraction expansion

$$R(s) = \sum_{i=1}^{r} \sum_{k_i=1}^{N_i} \frac{A_{i,k_i}}{(s+a_i)^{k_i}}$$

Also recall

$$\frac{t^{n-1}}{(n-1)!} e^{-at} u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+a)^n}, \quad \text{Re}\, s > -a$$

$$-\frac{t^{n-1}}{(n-1)!} e^{-at} u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+a)^n}, \quad \text{Re}\, s < -a$$

By linearity, $\mathcal{L}^{-1}\{R\}$ is a linear combination of terms of the above form, where signs are chosen according to the ROC.

Example

Consider a rational Laplace transform $X(s) = \frac{1}{(s+1)(s+2)^2}$

$$X(s) = \frac{1}{s+1} + \frac{-1}{s+2} + \frac{-1}{(s+2)^2}$$

Two poles at $s = -1$ and $s = -2$.

1. If ROC is $\mathrm{Re}\, s > -1$

$$x(t) = e^{-t}u(t) - (1+t)e^{-2t}u(t)$$

2. If ROC is $\mathrm{Re}\, s < -2$

$$x(t) = -e^{-t}u(-t) + (1+t)e^{-2t}u(-t)$$

3. If ROC is $-2 < \mathrm{Re}\, s < -1$

$$x(t) = -e^{-t}u(-t) - (1+t)e^{-2t}u(t)$$

Inverse Transform by Contour Integration

Suppose $X(s)$ has finitely many finite isolated singularities $s_1, \ldots, s_n$, and ROC is $-\infty \leq \sigma_1 < \operatorname{Re} s < \sigma_2 \leq +\infty$.

The inverse transform is

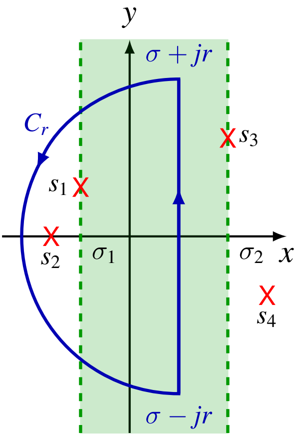
$$x(t) = \lim_{r \to \infty} \frac{1}{j2\pi} \int_{\sigma - jr}^{\sigma + jr} X(s) e^{st} ds$$

where $\sigma_1 < \sigma < \sigma_2$.

For $t > 0$, choose a large semicircle C_r that encloses all s_k with $\operatorname{Re} s \leq \sigma_1$.

If $X(s)$ satisfies[1] (a shifted and rotated version of) Jordan's Lemma (slide 6, Lecture 27), Residue Theorem implies

$$x(t) = \sum_{k: s_k \leq \sigma_1} \operatorname{Res}[X(s)e^{st}, s_k], \quad t > 0$$



[1] satisfied by proper rational functions

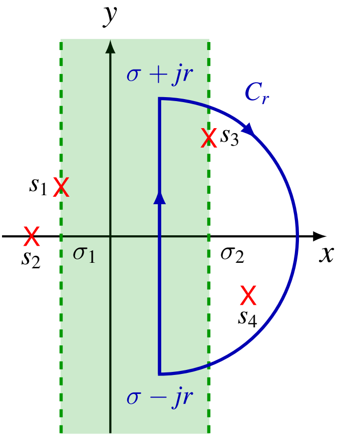
Inverse Transform by Contour Integration

Suppose $X(s)$ has finitely many finite isolated singularities $s_1, \ldots, s_n$, and ROC is $-\infty \leq \sigma_1 < \operatorname{Re} s < \sigma_2 \leq +\infty$.

The inverse transform is

$$x(t) = \lim_{r\to\infty} \frac{1}{j2\pi} \int_{\sigma-jr}^{\sigma+jr} X(s)e^{st}ds$$

where $\sigma_1 < \sigma < \sigma_2$.



For $t < 0$, choose a large semicircle C_r that encloses all s_k with $\operatorname{Re} s \geq \sigma_2$.

If $X(s)$ satisfies (a shifted and rotated version of) Jordan's Lemma, Residue Theorem implies

$$x(t) = -\sum_{k:s_k\geq\sigma_2} \operatorname{Res}[X(s)e^{st}, s_k], \quad t < 0$$

Caution. Negative sign due to negative orientation of contour.

Example

Consider a rational Laplace transform $X(s) = \frac{1}{(s+1)(s+2)^2}$

$$\mathrm{Res}[X(s)e^{st}, -1] = \left.\frac{e^{st}}{(s+2)^2}\right|_{s=-1} = e^{-t}$$

$$\mathrm{Res}[X(s)e^{st}, -2] = \left[\frac{d}{ds}\frac{e^{st}}{s+1}\right]_{s=-2} = -(1+t)e^{-2t}$$

1. If ROC is Re $s > -1$, then $\sigma_1 = -1, \sigma_2 = +\infty$.

   For $t > 0$,

   $$x(t) = \sum_{k:\text{Re}\, s_k \leq -1} \mathrm{Res}[X(s)e^{st}, s_k] = e^{-t} - (1+t)e^{-2t}$$

   For $t < 0$,

   $$x(t) = -\sum_{k:\text{Re}\, s_k \geq +\infty} \mathrm{Res}[X(s)e^{st}, s_k] = 0$$

Example (cont'd)

Consider a rational Laplace transform $X(s) = \frac{1}{(s+1)(s+2)^2}$

$$\mathrm{Res}[X(s)e^{st}, -1] = \left.\frac{e^{st}}{(s+2)^2}\right|_{s=-1} = e^{-t}$$

$$\mathrm{Res}[X(s)e^{st}, -2] = \left[\frac{d}{ds}\frac{e^{st}}{s+1}\right]_{s=-2} = -(1+t)e^{-2t}$$

2. If ROC is $\mathrm{Re}\, s < -2$, then $\sigma_1 = -\infty, \sigma_2 = -2$.

   For $t > 0$,

$$x(t) = \sum_{k:\mathrm{Re}\, s_k \leq -\infty} \mathrm{Res}[X(s)e^{st}, s_k] = 0$$

   For $t < 0$,

$$x(t) = -\sum_{k:\mathrm{Re}\, s_k \geq -2} \mathrm{Res}[X(s)e^{st}, s_k] = -e^{-t} + (1+t)e^{-2t}$$

Example (cont'd)

Consider a rational Laplace transform $X(s) = \frac{1}{(s+1)(s+2)^2}$

$$\mathrm{Res}[X(s)e^{st}, -1] = \left.\frac{e^{st}}{(s+2)^2}\right|_{s=-1} = e^{-t}$$

$$\mathrm{Res}[X(s)e^{st}, -2] = \left[\frac{d}{ds}\frac{e^{st}}{s+1}\right]_{s=-2} = -(1+t)e^{-2t}$$

3. If ROC is $-2 < \text{Re}\, s < -1$, then $\sigma_1 = -2, \sigma_2 = -1$.

   For $t > 0$,

$$x(t) = \sum_{k:\text{Re}\, s_k \leq -2} \mathrm{Res}[X(s)e^{st}, s_k] = -(1+t)e^{-2t}$$

   For $t < 0$,

$$x(t) = -\sum_{k:\text{Re}\, s_k \geq -1} \mathrm{Res}[X(s)e^{st}, s_k] = -e^{-t}$$

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Laplace Transform of Singularity Functions

We can also define Laplace transform for generalized functions.

In this course, we only consider the Laplace transforms of $\delta(t)$ and its derivatives. Formally,

$$\mathcal{L}\{\delta\} = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt = 1$$

$$\mathcal{L}\{\delta'\} = \int_{-\infty}^{\infty} \delta'(t) e^{-st} dt = -\frac{d}{dt} e^{-st}\Big|_{t=0} = s$$

$$\mathcal{L}\{\delta^{(k)}\} = \int_{-\infty}^{\infty} \delta^{(k)}(t) e^{-st} dt = (-1)^k \frac{d^k}{dt^k} e^{-st}\Big|_{t=0} = s^k$$

The Laplace transforms are defined for all $s \in \mathbb{C}$, and we say "ROC" $= \mathbb{C}$, though there is no convergence issue involved.

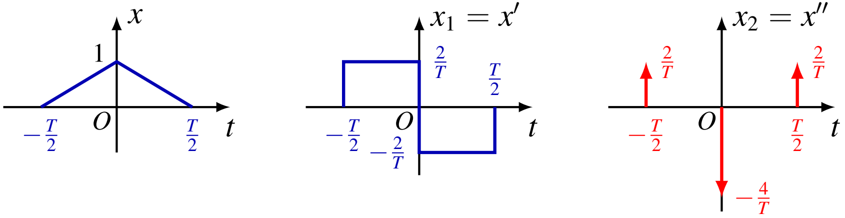
Laplace Transform of Singularity Functions

The properties of Laplace transforms discussed previously still holds for this generalization.

Example. The property of time differentiation is equivalent to the convolution property involving δ'.

$$\mathcal{L}\{x'\} = \mathcal{L}\{x * \delta'\} = \mathcal{L}\{x\}\mathcal{L}\{\delta'\} = s\mathcal{L}\{x\}$$

Example.



$$X_2(s) = -\frac{4}{T} + \frac{2}{T}e^{-s\frac{T}{2}} + \frac{2}{T}e^{s\frac{T}{2}} = \frac{8}{T}\sinh^2\left(\frac{sT}{4}\right) \implies X(s) = \frac{X_2(s)}{s^2}$$

Inverse Laplace Transform

Recall a rational function can be written as the sum of a polynomial and a proper rational function

$$R(s) = \sum_{k=1}^{n} a_k s^k + R_1(s)$$

Thus

$$\mathcal{L}^{-1}\{R\} = \sum_{k=1}^{n} a_k \delta^{(k)}(t) + \mathcal{L}^{-1}\{R_1\}$$

where $\mathcal{L}^{-1}\{R_1\}$ can be found by partial fraction expansion or contour integration.

Example. $X(s) = \dfrac{s(s+2)}{s+3} = s - 1 + \dfrac{3}{s+3}$ with ROC $\mathrm{Re}\, s > -3$, so

$$x(t) = \delta'(t) - \delta(t) + 3e^{-3t}u(t)$$

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CT System Function

Recall the response of a CT LTI system to the input $x(t)$ is

$$y(t) = (x * h)(t)$$

where h is the impulse response of the system.

If x and h have Laplace transforms, the convolution property implies

$$Y(s) = X(s)H(s)$$

in their common ROC[2].

If the ROC has a nonempty interior point, the system function (aka transfer function) $H(s)$ uniquely determines h and hence system properties through the inverse Laplace transform (this can be proved, but we will not do so).

[2] Actually ROAC for ordinary functions. For most signals in this course, ROC and ROAC coincide, so we will be sloppy.

Causality

Recall h is right-sided iff the ROC of $H(s)$ is a right half-plane,

$$\boxed{\textbf{causal} \implies \textbf{ROC is a right half-plane}}$$

Caution. The converse is **not** true.

Example. Consider $H(s) = \dfrac{e^s}{s+1}$ with ROC $\operatorname{Re} s > -1$. By the time-shift property,

$$h(t) = e^{-(t+1)}u(t+1)$$

which is not causal.

Causality

An LTI system with rational system function $H(s)$ is causal iff the ROC is the right half-plane to the rightmost pole.

Proof. Recall for a rational system function,

$$H(s) = \sum_{k=1}^{n} a_k s^k + \sum_{i=1}^{r} \sum_{k_i=1}^{N_i} \frac{A_{i,k_i}}{(s+a_i)^{k_i}}, \quad \text{Re}\, s > \max_i \text{Re}\,(-a_i)$$

so

$$h(t) = \sum_{k=1}^{n} a_k \delta^{(k)}(t) + \sum_{i=1}^{r} \sum_{k_i=1}^{N_i} \frac{A_{i,k_i}}{(k_i-1)!} t^{k_i-1} e^{-a_i t} u(t)$$

which is causal. (Can also prove by contour integration.)

Example. $h(t) = e^{-t}u(t) \xleftrightarrow{\mathcal{L}} H(s) = \frac{1}{s+1}$ with $\text{Re}\, s > -1$, causal.

Example. $h(t) = e^{-|t|} \xleftrightarrow{\mathcal{L}} H(s) = \frac{-2}{s^2-1}$ with $-1 < \text{Re}\, s < 1$, noncausal.

Stability

Recall an LTI system is stable iff its impulse response $h \in L_1$, i.e.

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

i.e. $H(s)$ converges absolutely on the imaginary axis $\mathrm{Re}\, s = 0$, so its ROC $-\infty \le \sigma_1 < \mathrm{Re}\, s < \sigma_2 \le \infty$ must satisfy $\sigma_1 < 0 < \sigma_2$.

stable $\iff$ ROC includes the imaginary axis

A **causal** LTI system with rational system function $H(s)$ is stable iff all its poles have negative real parts.

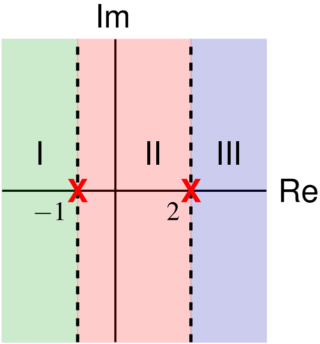
Example. A causal system with $H(s) = \frac{1}{s+a}$ is stable iff $\mathrm{Re}\, a > 0$

Example. A system with $H(s) = \frac{1}{s+a}$ where $\mathrm{Re}\, a < 0$ and ROC $\mathrm{Re}\, s < -\mathrm{Re}\, a$ is also stable, but it is noncausal.

Example

Consider the system function

$$H(s) = \frac{1}{s-2} - \frac{1}{s+1}$$



There are two poles $p_1 = -1$ and $p_2 = 2$.

1. $\mathrm{Re}\, s < -1$, noncausal, unstable

$$h_1(t) = -e^{2t}u(-t) + e^{-t}u(-t)$$

2. $-1 < \mathrm{Re}\, s < 2$, noncausal, stable

$$h_2(t) = -e^{2t}u(-t) - e^{-t}u(t)$$

3. $\mathrm{Re}\, s > 2$, causal, unstable

$$h_3(t) = e^{2t}u(t) - e^{-t}u(t)$$

Linear Constant-coefficient ODE

LTI system with input and output related by

$$\sum_{k=0}^{N} a_k \frac{d^k y}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x}{dt^k}$$

Take Laplace transform of both sides

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

so

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

System function is always rational

ODE does not specify ROC! Need additional conditions (e.g. stability, causality) to determine $h(t)$.

Example

Consider LTI system with input and output related by

$$y'(t) + 3y(t) = x'(t) + x(t)$$

System function

$$H(s) = \frac{s+1}{s+3} = 1 - \frac{2}{s+3}$$

Two possibilities for ROC: Re $s > -3$ and Re $s < -3$

1. If Re $s > -3$, causal and stable,

$$h(t) = \delta(t) - 2e^{-3t}u(t)$$

2. If Re $s < -3$, anticausal and unstable,

$$h(t) = \delta(t) + 2e^{-3t}u(-t)$$

Example (cont'd)

Consider LTI system with input and output related by

$$y'(t) + 3y(t) = x'(t) + x(t)$$

If we use Fourier transform, then frequency response is

$$H(j\omega) = \frac{j\omega + 1}{j\omega + 3} = 1 - \frac{2}{j\omega + 3}$$

and

$$h(t) = \delta(t) - 2e^{-3t}u(t)$$

Why only one possibility?

- Fourier transform method assumes stability, requiring that ROC of $H(s)$ contain the imaginary axis, so Re $s > -3$
- In general, not applicable to unstable systems

Example (cont'd)

Find response to $x(t) = e^{-4t}u(t)$.

$$X(s) = \frac{1}{s+4}, \quad \text{Re}\, s > -4$$

Laplace transform for response

$$Y(s) = H(s)X(s) = \frac{s+1}{s+3} \cdot \frac{1}{s+4} = \frac{-2}{s+3} + \frac{3}{s+4}$$

Two possible ROCs

1. If $\text{Re}\, s > -3$,

$$y(t) = -2e^{-3t}u(t) + 3e^{-4t}u(t)$$

2. If $-4 < \text{Re}\, s < -3$,

$$y(t) = 2e^{-3t}u(-5) + 3e^{-4t}u(t)$$

Example (cont'd)

Find response to $x(t) = e^{-3t}u(t)$.

$$X(s) = \frac{1}{s+3}, \quad \text{Re}\, s > -3$$

Laplace transform for response

$$Y(s) = H(s)X(s) = \frac{s+1}{s+3} \cdot \frac{1}{s+3} = \frac{1}{s+3} + \frac{-2}{(s+3)^2}$$

Only one possible ROC $\text{Re}\, s > -3$

$$y(t) = (1-2t)e^{-3t}u(t)$$

For the anticausal and unstable system,

$$h(t) = \delta(t) + 2e^{-3t}u(-t)$$

Can verify directly $x * h$ is not well-defined.

Example

The response of an LTI system to the input $x(t) = e^{-3t}u(t)$ is

$$y(t) = (e^{-t} - e^{-2t})u(t)$$

System function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{1}{s+1} - \frac{1}{s+2}}{\frac{1}{s+3}} = \frac{s+3}{s^2+3s+2}, \quad \mathrm{Re}\, s > -1$$

By partial fraction expansion

$$H(s) = \frac{2}{s+1} - \frac{1}{s+2}, \ \mathrm{Re}\, s > -1 \implies h(t) = (2e^{-t} - e^{-2t})u(t)$$

Can be described by the following ODE

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \frac{dx}{dt} + 3x$$

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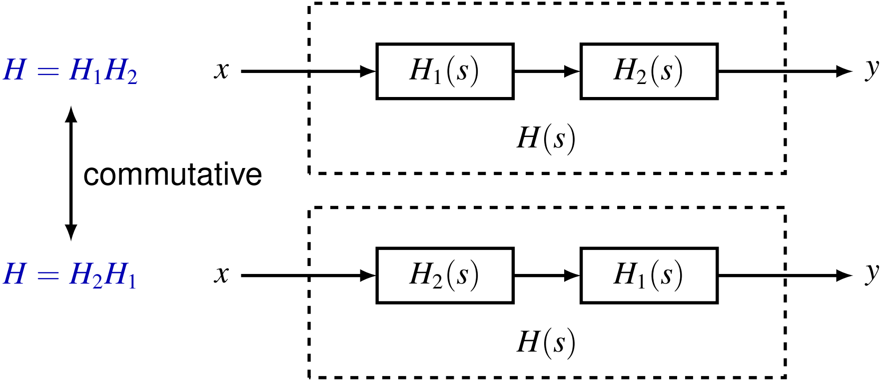
System Interconnections

LTI systems in series connection

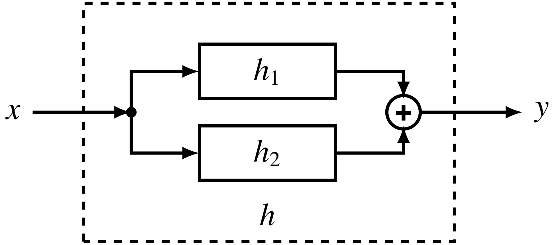
$$Y = (XH_1)H_2 = X(H_1H_2) = (XH_2)H_1$$



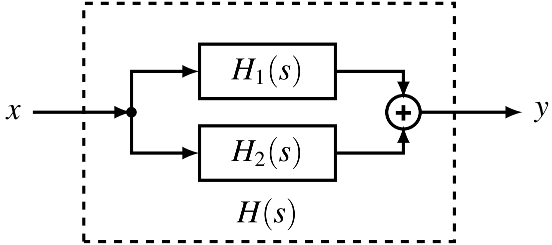
System Interconnections

LTI systems in systems in parallel connection

$h = h_1 + h_2$



$H = H_1 + H_2$

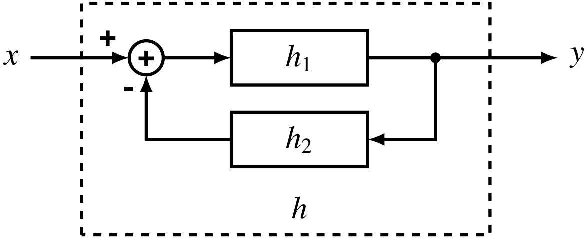


System Interconnections

LTI systems in systems in feedback connection

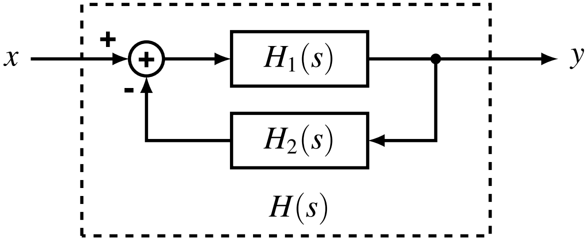
$$y = h_1 * (x - h_2 * y)$$

$$h = ?$$



$$Y = H_1 X - H_1 H_2 Y$$

$$H = \frac{Y}{X} = \frac{H_1}{1 + H_1 H_2}$$



Example

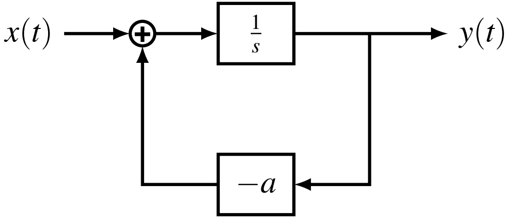
Causal LTI systems with system function

$$H(s) = \frac{1}{s+a}, \quad \text{Re}\, s > -\text{Re}\, a$$

Equivalent description by ODE

$$y'(t) + ay(t) = x(t)$$

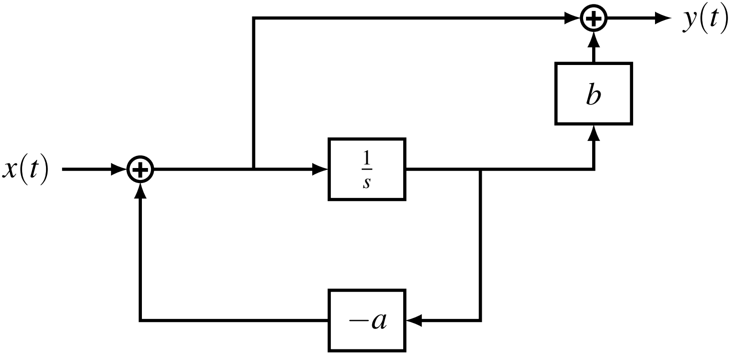
with initial rest condition.



Example

Causal LTI systems with system function

$$H(s) = \frac{s+b}{s+a} = \left(\frac{1}{s+a}\right)(s+b), \quad \text{Re}\, s > -\text{Re}\, a$$



Example

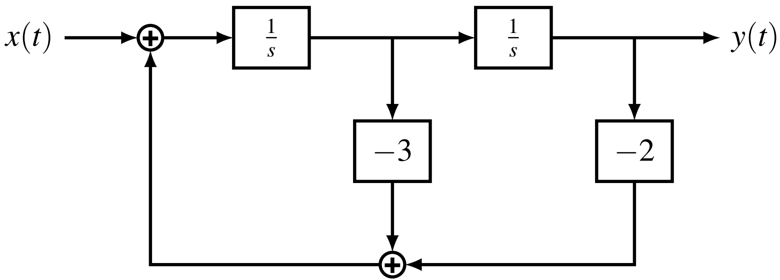
Causal LTI systems with system function

$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s^2+3s+2}$$

Equivalent description by different equation

$$y''(t) + 3y'(t) + 2y(t) = x(t)$$

Direct form

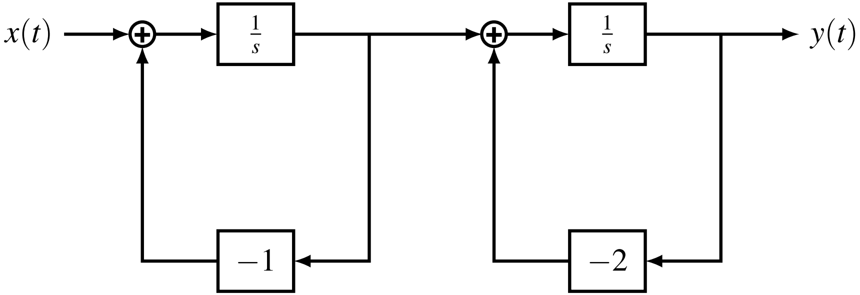


Example (cont'd)

Causal LTI systems with system function

$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} \cdot \frac{1}{s+2}$$

Cascade form



Example (cont'd)

Causal LTI systems with system function

$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

Parallel form