EI331 Signals and Systems

Lecture 4

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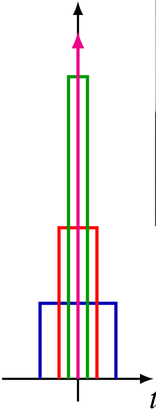
CT Unit Impulse Function

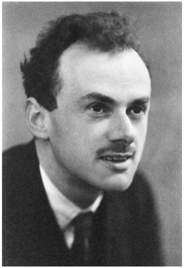
Also called Dirac delta function or δ function

$$\delta(t) = \lim_{\Delta \to 0} r_\Delta(t)$$

where

$$r_\Delta(t) = \frac{u(t + \frac{\Delta}{2}) - u(t - \frac{\Delta}{2})}{\Delta}$$





Paul Dirac
(from Wikipedia)

Idealization for quantities of very large magnitude but very small duration (e.g. impulse force) or spatial span (e.g. point mass/charge)

By usual calculus

$$\lim_{\Delta \to 0} r_\Delta(t) = \begin{cases} 0, & t \neq 0 \\ +\infty, & t = 0 \end{cases}$$

not properly defined at $t = 0$

Analogy with Construction of Real Numbers

Real numbers

- defined by (equivalence classes) of Cauchy sequences in $\mathbb{Q}$
- arithmetic: $x = \{x_n\} \subset \mathbb{Q}$, $y = \{y_n\} \subset \mathbb{Q}$

$$x + y \triangleq \{x_n + y_n\}, \quad xy \triangleq \{x_n y_n\}$$

Unit impulse

- **not** ordinary function
- singularity (generalized) function
- defined by "convergent" sequence of short pulses

Interpretation of Limit

Idea. Define δ in terms of integration

For any $\phi(t)$ **continuous at $t = 0$**,

$$\boxed{\int_{\mathbb{R}} \delta(t)\phi(t)dt \triangleq \lim_{\Delta\to 0} \int_{\mathbb{R}} r_{\Delta}(t)\phi(t)dt}$$

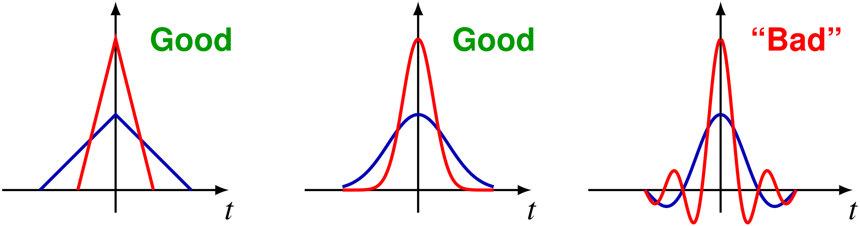
By continuity of ϕ,

$$\int_{\mathbb{R}} r_{\Delta}(t)\phi(t)dt = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} \phi(t)dt \to \phi(0)$$

Sampling property $\int_{-\infty}^{\infty} \delta(t)\phi(t)dt = \phi(0)$

Other Approximations

Can define δ as limit of other functions.



$$g_\Delta(t) = \frac{1}{\sqrt{2\pi}\Delta} e^{-\frac{t^2}{2\Delta^2}} \qquad D_\Delta(t) = \frac{\sin(\frac{\pi t}{\Delta})}{\pi t}$$

Family $\{K_\Delta(t)\}_{\Delta>0}$ called good kernels or approximation to the identity if

1. For all $\Delta > 0$, $\int_{-\infty}^{\infty} K_\Delta(t)dt = 1$
2. For some $M > 0$ and all $\Delta > 0$, $\int_{-\infty}^{\infty} |K_\Delta(t)|dt < M$
3. For every $\epsilon > 0$, $\lim\limits_{\Delta\to 0} \int_{|t|>\epsilon} |K_\Delta(t)|dx = 0$

Properties of Unit Impulse Function

Unit "area"

$$\int_{\mathbb{R}} \delta(\tau) d\tau = 1$$

Proof. Apply sampling property to $\phi(t) = 1$.

Relation to $u(t)$

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \triangleq \int_{\mathbb{R}} \delta(\tau) u(t-\tau) d\tau, \qquad \delta(t) = \frac{d}{dt} u(t)$$

Proof. For integration, apply sampling property. Note $u(t-\tau)$ is continuous at $\tau = 0$ for $t \neq 0$. For differentiation, $u'(t) = \lim_{\Delta \to 0} r_\Delta(t)$ (will come back later).

In general, $$\int_a^b f(\tau) d\tau \triangleq \int_{\mathbb{R}} f(\tau)[u(\tau - a) - u(\tau - b)] d\tau$$

Transformations of Unit Impulse

Usual rules for change of variables hold

Time scaling

$$\int_{\mathbb{R}} \delta(at)\phi(t)dt \triangleq \int_{\mathbb{R}} \delta(t)\phi\left(\frac{t}{a}\right)\frac{dt}{|a|} \implies \delta(at) = \frac{1}{|a|}\delta(t)$$

Time reversal

$$\int_{\mathbb{R}} \delta(-t)\phi(t)dt \triangleq \int_{\mathbb{R}} \delta(t)\phi(-t)dt \implies \delta(-t) = \delta(t)$$

Time shift (general sampling property)

$$\int_{\mathbb{R}} \delta(t-a)\phi(t)dt \triangleq \int_{\mathbb{R}} \delta(t)\phi(t+a)dt = \phi(a)$$

Multiplication and Sampling Property

Multiplication by ordinary function

$$\int_{\mathbb{R}} [x(t)\delta(t)]\phi(t)dt \triangleq \int_{\mathbb{R}} \delta(t)[x(t)\phi(t)]dt = x(0)\phi(0)$$

Sampling property

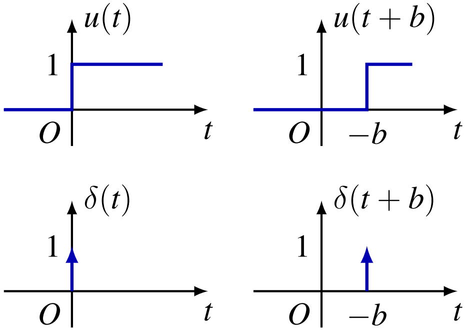
$$x\delta = x(0)\delta, \quad \text{or} \quad x(t)\delta(t) = x(0)\delta(t)$$
$$x\tau_a\delta = x(a)\tau_a\delta, \quad \text{or} \quad x(t)\delta(t-a) = x(a)\delta(t-a)$$

Just a restatement of the following

$$\int_{\mathbb{R}} [x(t)\delta(t-a)]\phi(t)dt = x(a)\phi(a) = \int_{\mathbb{R}} [x(a)\delta(t-a)]\phi(t)dt$$

Statements about δ always interpreted this way!

Derivative of $u(at+b)$



Chain rule holds

$$\frac{d}{dt}u(at+b) = a\delta(at+b)$$

“Proof”.

1. $\frac{d}{dt}u(t+b) = \delta(t+b)$
2. $a > 0 \implies u(at+b) = u(t+b/a)$
$$\frac{d}{dt}u(at+b) = \frac{d}{dt}u\left(t+\frac{b}{a}\right) = \delta\left(t+\frac{b}{a}\right) = a\delta(at+b)$$
3. $a < 0 \implies u(at+b) = 1 - u(t+b/a)$
$$\frac{d}{dt}u(at+b) = -\frac{d}{dt}u\left(t+\frac{b}{a}\right) = -\delta\left(t+\frac{b}{a}\right) = -|a|\delta(at+b)$$

Derivative of $x(t)u(t)$

Leibniz rule holds

For differentiable x,

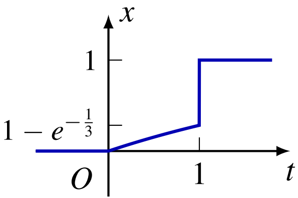
$$\begin{aligned}[x(t)u(t)]' &= x'(t)u(t) + x(t)u'(t) \\ &= x'(t)u(t) + x(t)\delta(t) \\ &= \underbrace{x'(t)u(t)}_{\text{ordinary derivative}} + \underbrace{x(0)\delta(t)}_{\text{derivative at discontinuity}}\end{aligned}$$

Will see later general procedure for taking derivatives.

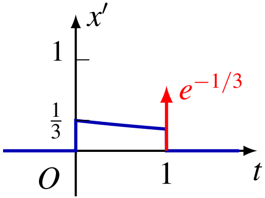
Functions with Jump Discontinuities

Example.

$$x(t) = (1 - e^{-\frac{1}{3}t})[u(t) - u(t-1)] + u(t-1)$$
$$= \begin{cases} 0, & t < 0 \\ 1 - e^{-t/3}, & 0 < t < 1 \\ 1, & t > 1 \end{cases}$$



$$x'(t) = \frac{1}{3}e^{-\frac{1}{3}t}[u(t) - u(t-1)] + e^{-\frac{1}{3}}\delta(t-1)$$



1. impulse at each discontinuity
2. impulse size equal to jump size

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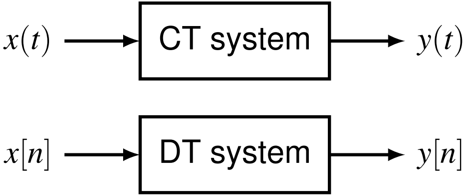
Systems

A system takes some input and produces some output.

Mathematically, $y = T(x)$ for some operator T.



Example. Balance of bank account.

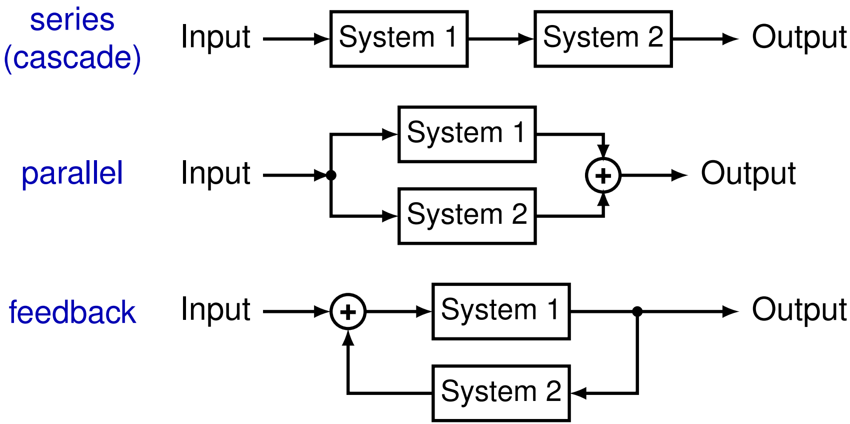
- Input $x[n]$: net deposit on n-th day
- Output $y[n]$: balance at end of n-th day
- Input-output relation

$$y[n] = (1 + r)y[n - 1] + x[n], \qquad r \text{ interest rate}$$

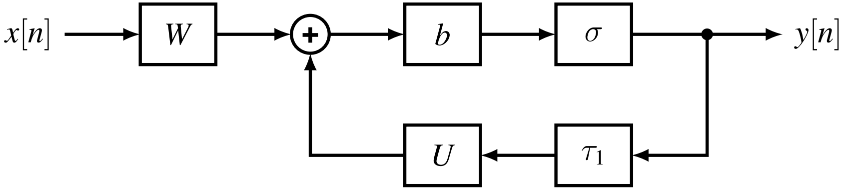
Interconnections of Systems

- Complex systems built from interconnected subsystems
- Scope of subsystem depends on level of abstraction

Basic Types of Interconnections



Example



Subsystems

- W: $y[n] = Wx[n]$
- σ: $y[n] = \sigma(x[n])$
- τ_1: $y[n] = x[n-1]$
- b: $y[n] = x[n] + b$
- U: $y[n] = Ux[n]$

Composite system (Recurrent neural network)

$$y[n] = \sigma(Wx[n] + Uy[n-1] + b)$$

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Memory

System is memoryless if output depends only on **input** at the same time.

Example. Identity system $y = I(x) = x$

$$y(t) = x(t), \qquad y[n] = x[n]$$

Example. Multiplication by known function $y = ax$

$$y(t) = a(t)x(t), \qquad y[n] = a[n]x[n]$$

- resistor: $v(t) = Ri(t)$
- $y(t) = \sin(t+1)x(t)$ memoryless?

  Yes ! $a(t) = \sin(t+1)$ **not** part of input !

Example. Can take complicated form

$$y(t) = x^3(t) - 2x(t) + e^{x(t)} + \sin(\cos(x(t))) + \cos(t+1)$$

Memory

System has memory (non-memoryless) if not memoryless

Example. Time shift $y = \tau_a x$ for $a \neq 0$

$$y(t) = x(t-a), \qquad y[n] = x[n-a]$$

- $a > 0$: output depends on past input
- $a < 0$: output depends on **future** input **("memory" !)**

Example. Integrator and accumulator

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau, \qquad y[n] = \sum_{k=-\infty}^{n} x[k]$$

- capacitor (used in DRAM !): $v(t) = \int_{-\infty}^{t} C^{-1} i(\tau) d\tau$

Example. Differentiator

$$y(t) = \frac{d}{dt} x(t) = \lim_{a \to 0} \frac{x(t+a) - x(t)}{a}$$

Invertibility

System is invertible if distinct inputs yield distinct outputs

Mathematically, system operator T is injective, i.e.

$$\forall x_1, x_2, \quad x_1 \neq x_2 \implies T(x_1) \neq T(x_2)$$

System is non-invertible if not invertible, i.e.

$$\exists x_1, x_2, \quad x_1 \neq x_2 \text{ but } T(x_1) = T(x_2)$$

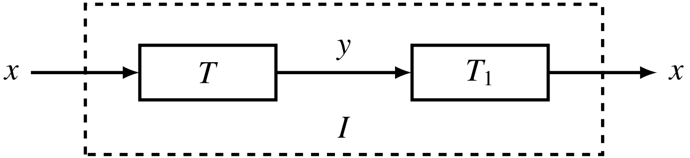
Example. Multiplication by known function $y = ax$

- invertible if $a(t) \neq 0$ for all t, e.g. $y(t) = e^t x(t)$
- non-invertible if $a(t) = 0$ for some t, e.g. $y(t) = u(t)x(t)$

Example. $y(t) = x^2(t)$ is non-invertible, since $x^2 = (-x)^2$

Invertibility

System T_1 is inverse system of system T if cascade of T and T_1 forms identity system, i.e. $T_1 \circ T = I$



System is invertible iff it has inverse system.

Example. $y(t) = 2x(t)$ has inverse system $y(t) = \frac{1}{2}x(t)$

Example. Inverse system of accumulator $y[n] = \sum_{k=-\infty}^{n} x[k]$ is first difference $y[n] = x[n] - x[n-1]$

Caution. Not symmetric. First difference is non-invertible.

Causality

System is causal if output at **any** time depends only on input values up to that time

Also called nonanticipative, i.e. output at **any** time does not depend on (anticipate) future input values

Example. First difference

- backward difference is causal $y[n] = x[n] - x[n-1]$
- forward difference is noncausal $y[n] = x[n+1] - x[n]$

Example. Moving average is noncausal

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{M} x[n-k], \quad M \geq 1$$

Example. $y(t) = \sin(t+1)x(t)$ is causal

Causality

- For causal systems, identical inputs up to some time yield identical outputs up to the same time

$$x_1(t) = x_2(t) \text{ for } t \le t_0 \implies (Tx_1)(t) = (Tx_2)(t) \text{ for } t \le t_0$$

$$x_1[n] = x_2[n] \text{ for } n \le n_0 \implies (Tx_1)[n] = (Tx_2)[n] \text{ for } n \le n_0$$

- Causality is important when t (or n) is time
  - real-time physical systems are causal, cause before effect
  - non-real-time systems can be noncausal, e.g. postprocessing of recorded signals

$$\underbrace{y[n] = \frac{1}{2M+1} \sum_{k=-M}^{M} x[n-k]}_{\text{noncausal}}, \quad \text{vs.} \quad \underbrace{y[n] = \frac{1}{M+1} \sum_{k=0}^{M} x[n-k]}_{\text{causal}}$$

  - not meaningful if t (or n) is spatial variable

Stability

Many different notions of stability.

System is bounded-input bounded-output (BIBO) stable if outputs are bounded for all bounded inputs.

- Signal x is bounded if for some constant B

$$|x(t)| \leq B, \quad \forall t, \qquad\qquad |x[n]| \leq B, \quad \forall n$$

Or $\|x\|_\infty = \sup_t |x(t)| < \infty$, $\quad \|x\|_\infty = \sup_n |x[n]| < \infty$

- System is BIBO stable if

$$\|x\|_\infty < \infty \implies \|T(x)\|_\infty < \infty$$

Stability

Example. Exponentiation $y(t) = e^{x(t)}$ is stable

$$|x(t)| \le B \implies |y(t)| \le e^B$$

Example. Accumulator $y[n] = \sum_{k=-\infty}^{n} x[k]$ is unstable

$x[n] = u[n]$ bounded, but $y[n] = (n+1)u[n]$ unbounded

Example. First difference $y[n] = x[n] - x[n-1]$ is stable

$$|x[n]| \le B \implies |y[n]| \le |x[n]| + |x[n-1|] \le 2B$$

Example. Differentiator $y(t) = \frac{d}{dt}x(t)$ is unstable

$$|\sin(t^2)| \le 1, \text{ but } \left|\frac{d}{dt}\sin(t^2)\right| = |2t\cos(t^2)| \text{ unbounded}$$

Time Invariance

System is time invariant if time shift in input results in identical time shift in output

- conceptually, system behavior independent of time of usage
- mathematically

$$T \circ \tau_a = \tau_a \circ T$$

$$\begin{tikzcd} x(t) \arrow[r, "T"] \arrow[d, "\tau_a"] & y(t) \arrow[d, "\tau_a"] \\ x(t-a) \arrow[r, "T"] & y(t-a) \end{tikzcd} \qquad \begin{tikzcd} x[n] \arrow[r, "T"] \arrow[d, "\tau_a"] & y[n] \arrow[d, "\tau_a"] \\ x[n-a] \arrow[r, "T"] & y[n-a] \end{tikzcd}$$

Time Invariance

Example. The following systems are time-invariant

1. $y(t) = \sin(x(t))$
2. $y[n] = x^2[n]$
3. $y[n] = x[n] - x[n-1]$

Example. The following systems are time-varying

1. $y[n] = nx[n]$
2. $y(t) = x(t)\cos(\omega t)$ amplitude modulation
3. $y(t) = x(-t)$
4. $y(t) = x(2t)$

Example. Time-invariant systems have periodic outputs for periodic inputs

Linearity

System is linear if it has superposition property

$$T(a_1x_1 + a_2x_2) = a_1T(x_1) + a_2T(x_2)$$

or, equivalently, if it is additive and homogeneous,

1. additivity
$$T(x_1 + x_2) = T(x_1) + T(x_2)$$
2. homogeneity
$$T(ax) = aT(x)$$

Example. $y(t) = tx(t)$ is linear
Example. $y(t) = x^2(t)$ is nonlinear
Example. $y(t) = \sin(x(t))$ is nonlinear
Example. $y(t) = x(\sin t)$ is linear

Linearity

Why care about linear systems?

1. accurate models for many systems
   - resistor, capacity, Newton's law, etc
2. mathematical tractability, many powerful tools
3. linearization of nonlinear systems
   - "small signal" perturbation around "operating point"

$$y(t) = f(x(t)) \implies \Delta y(t) \approx f'(x_0(t))\Delta x(t)$$

     where $\Delta y(t) = y(t) - f(x_0(t))$, $\Delta x(t) = x(t) - x_0(t)$

   - provides insights for behavior of nonlinear system

Wide Neural Networks of Any Depth Evolve as Linear Models Under Gradient Descent

Jaehoon Lee, Lechao Xiao, Samuel S. Schoenholz, Yasaman Bahri, Jascha Sohl-Dickstein, Jeffrey Pennington

(Submitted on 18 Feb 2019)

Linearity

General superposition property

$$T\left(\sum_k a_k x_k\right) = \sum_k a_k T(x_k)$$

1. finitely many terms: by induction
2. infinitely many terms: need continuity property, i.e.

$$T\left(\lim_{k\to\infty} x_k\right) = \lim_{k\to\infty} T(x_k)$$

Will (implicitly) assume continuity in this course.

Linearity

Zero-in zero-out property

For linear system

$$T(0) = 0$$

where 0 is zero signal, i.e. $x(t) = 0, \forall t$ or $x[n] = 0, \forall n$

$T(0)$ called zero-input response of system

Proof. Use homogeneity.

Example. $y(t) = 2x(t) + 1$ is nonlinear **(!)** since $T(0) = 1$

T is incrementally linear if $\tilde{T}(x) = T(x) - T(0)$ is linear