EI331 Signals and Systems

Lecture 6

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Contents

Representation of CT Signals by Impulses

Sifting property of CT unit impulse

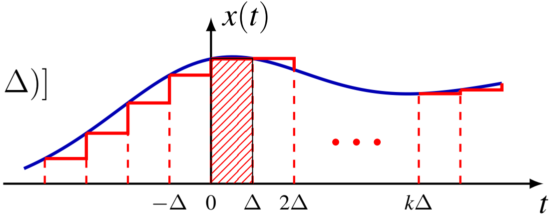
$$x(t) = \int_{\mathbb{R}} x(a)\delta(t-a)da$$

Interpreted as limit as $\Delta \to 0$ of

$$\hat{x}_\Delta(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)p_\Delta(t-k\Delta)\Delta$$

where

$$p_\Delta(t) = \frac{1}{\Delta}[u(t) - u(t-\Delta)]$$



CT Linear Systems

Response of linear system

$$\hat{y}_\Delta = T(\hat{x}_\Delta) = T\left(\sum_{k=-\infty}^{\infty} x(k\Delta)\tau_{k\Delta}p_\Delta\Delta\right)$$

$$= \sum_{k=-\infty}^{\infty} x(k\Delta)T(\tau_{k\Delta}p_\Delta)\Delta = \sum_{k=-\infty}^{\infty} x(k\Delta)\hat{h}_{k\Delta}\Delta$$

where $\hat{h}_{k\Delta} = T(\tau_{k\Delta}p_\Delta)$ is response to shifted pulse $\tau_{k\Delta}p_\Delta$.

In the limit $\Delta \to 0$,

- $\hat{x}_\Delta \to x$ and $\hat{y}_\Delta \to y = T(x)$
- for $k\Delta \to a$, have $\tau_{k\Delta}p_\Delta \to \delta_a$, expect $\hat{h}_{k\Delta} \to h_a = T(\delta_a)$

$$y = \int_{\mathbb{R}} x(a)h_a da, \quad \text{or} \quad y(t) = \int_{\mathbb{R}} x(a)h_a(t)da$$

CT Linear Time-invariant (LTI) Systems

Unit impulse response[1]

$$h = h_0 = T(\delta)$$

$$\text{time invariance} \implies h_a = T(\delta_a) = \tau_a(T(\delta)) = \tau_a h$$

Response of LTI system – Convolution integral

$$\boxed{y(t) = \int_{\mathbb{R}} x(\tau)h(t-\tau)d\tau, \quad \forall t \in \mathbb{R}}$$

LTI system is fully characterized by unit impulse response!

Conversely, given h, system $T(x)(t) \triangleq \int_{\mathbb{R}} x(\tau)h(t-\tau)d\tau$ is LTI

[1] For proof of existence, see Theorem 2 of VI.3 in Kôsaku Yosida.

Impulse Responses of Simple LTI Systems

Identity

$$h(t) = \delta(t)$$

Scaler multiplication

$$h(t) = K\delta(t)$$

Time shift

$$h(t) = \delta_a(t) \triangleq \delta(t-a)$$

Integrator

$$h(t) = \int_{-\infty}^{t} \delta(\tau) d\tau = u(t)$$

Differentiator

$$h(t) = \delta'(t) \quad \text{(to be defined)}$$

Convolution

$$(x_1 * x_2)(t) = \int_{\mathbb{R}} x_1(\tau)x_2(t-\tau)d\tau, \quad \forall t \in \mathbb{R}$$

Not always defined for arbitrary x_1 and x_2

Example. For $x_1(t) = u(t) = x_2(-t)$, integral divergent for all t.

Sufficient conditions for absolute convergence

1. Either x_1 or x_2 has compact support $\operatorname{supp} x = \overline{\{t : x(t) \neq 0\}}$, i.e. x_1 or x_2 vanishes outside finite interval.
2. x_1, x_2 both right-sided (or left-sided), i.e. $x_i(t) = 0$ for $t \leq t_i$ (or $t \geq t_i$), $\forall i \implies x_1 * x_2$ also right-sided (or left-sided)

Convolution

Sufficient conditions for absolute convergence (cont'd)

3. One of x_1 and x_2 has finite L_1 norm and the other finite L_p norm for $1 \leq p \leq \infty$, where L_p norm [2]

$$\|x\|_p \triangleq \begin{cases} \left(\int_{\mathbb{R}} |x(t)|^p dt \right)^{1/p}, & \text{if } 1 \leq p < \infty \\ \sup_{t \in \mathbb{R}} |x(t)|, & \text{if } p = \infty. \end{cases}$$

   If $\|x_1\|_1 < \infty$, then $\|x_1 * x_2\|_p \leq \|x_1\|_1 \cdot \|x_2\|_p$.

4. $\|x_1\|_p < \infty$ and $\|x_2\|_q < \infty$ for $1 \leq p, q \leq \infty$ and $p^{-1} + q^{-1} = 1$. In this case, $\|x_1 * x_2\|_\infty \leq \|x_1\|_p \cdot \|x_2\|_q$.

[2] More precisely, $\|x\|_\infty = \sup\{B \geq 0 : |x(t)| \leq B \text{ for almost every } t\}$

Calculation of Convolution

1. Plot both x_1 and x_2 as functions of τ, i.e. $x_1(\tau)$, $x_2(\tau)$
2. Reverse $x_2(\tau)$ to obtain $x_2(-\tau)$
3. Given t, shift $x_2(-\tau)$ by t to obtain $x_2(t-\tau)$
4. Multiply $x_1(\tau)$ and $x_2(t-\tau)$ pointwise to obtain $g_t(\tau) = x_1(\tau)x_2(t-\tau)$
5. Integrate g_t over τ to obtain $(x_1 * x_2)(t)$, i.e. $$(x_1 * x_2)(t) = \int_{\mathbb{R}} g_t(\tau) d\tau$$
6. Repeat 1-5 for each t

Convolution

Example. Let $x(t) = e^{-at}u(t)$ and $h(t) = u(t)$ with $a > 0$.

For $t < 0$,

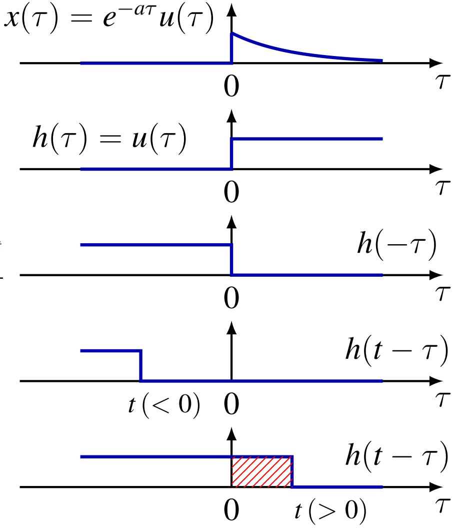
$$(x * h)(t) = 0$$

For $t \geq 0$,

$$(x*h)(t) = \int_0^t e^{-a\tau} d\tau = \frac{1 - e^{-at}}{a}$$

Thus

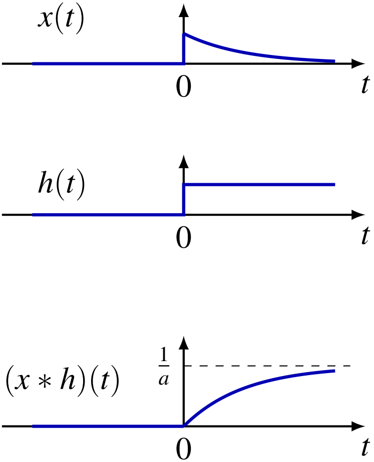
$$(x * h)(t) = \left(\frac{1 - e^{-at}}{a}\right) u(t)$$



Convolution

Example. Let $x(t) = e^{-at}u(t)$ and $h(t) = u(t)$ with $a > 0$.

$$
\begin{aligned}
(x * h)(t) &= \int_{\mathbb{R}} x(\tau)h(t-\tau)d\tau \\
&= \int_{\mathbb{R}} e^{-a\tau}u(\tau)u(t-\tau)d\tau \\
&= \int_{0\leq\tau\leq t} e^{-a\tau}d\tau \\
&= u(t)\int_0^t e^{-a\tau}d\tau \\
&= \left(\frac{1-e^{-at}}{a}\right)u(t)
\end{aligned}
$$

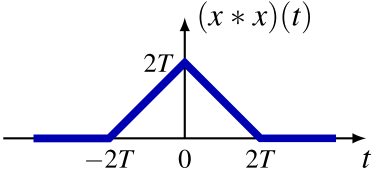


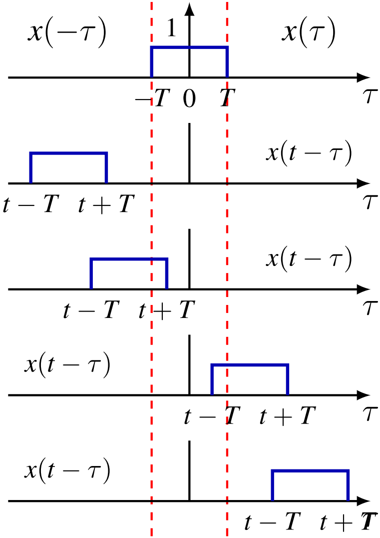
Also true for $a < 0$

Convolution

Example. Compute $x * x$, where $x(t) = u(t+T) - u(t-T)$. .

$$(x*x)(t) = \begin{cases} 0, & t < -2T \\ t + 2T, & -2T \le t < 0 \\ 2T - t, & 0 \le t < 2T \\ 0, & t \ge 2T \end{cases}$$





Convolution

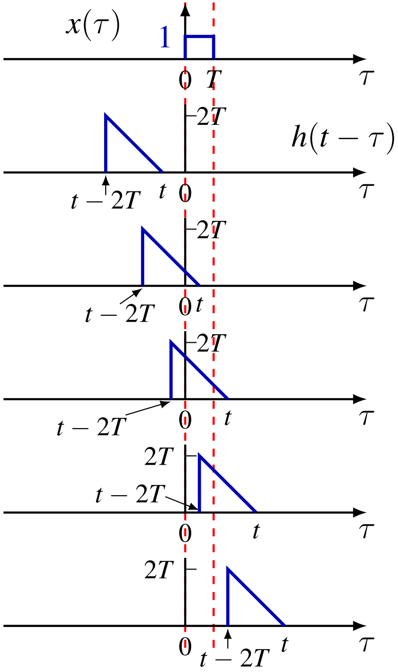
Example. Let

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$

$$h(t) = \begin{cases} t, & 0 \le t \le 2T \\ 0, & \text{otherwise} \end{cases}$$

Five cases

1. $t < 0$
2. $0 \le t \le T$
3. $T < t \le 2T$
4. $2T < t \le 3T$
5. $t > 3T$



Identity Element

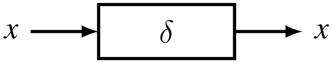
Recall sampling property of δ

$$x(t) = \int_{\mathbb{R}} x(\tau)\delta(t-\tau)d\tau, \quad \forall t \in \mathbb{R}$$

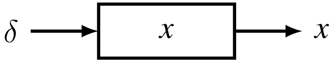
δ identity element for convolution

$$x = x * \delta = \delta * x$$

$x = x * \delta$



$x = \delta * x$



Properties of Convolution

Commutativity

$$x_1 * x_2 = x_2 * x_1$$

Bilinearity

$$\left(\sum_i a_i x_{1i}\right) * \left(\sum_j b_j x_{2j}\right) = \sum_i \sum_j a_i b_j (x_{1i} * x_{2j})$$

Associativity

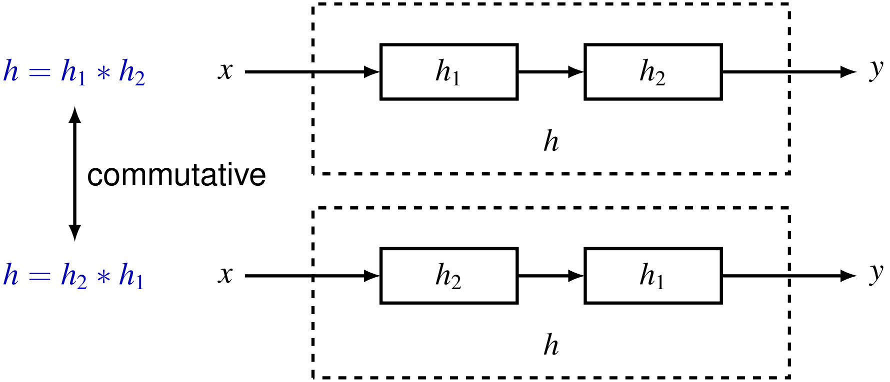
$$x_1 * x_2 * x_3 = (x_1 * x_2) * x_3 = x_1 * (x_2 * x_3)$$

Time shift

$$(\tau_a x_1) * (\tau_b x_2) = \tau_{a+b}(x_1 * x_2)$$

Associative Law

$$x_1 * (x_2 * x_3) = (x_1 * x_2) * x_3$$



Order of processing **usually** not important for LTI systems

Associative Law

Example. $x_1(t) = 1$, $x_2(t) = u(t)$, $x_3(t) = \delta'(t)$ (defined later)

1. $(x_2 * x_3)(t) = \delta(t)$, so $x_1 * (x_2 * x_3) = 1$
2. $x_1 * x_2$ and $(x_1 * x_2) * x_3$ undefined!
3. $x_1 * x_3 = 0$, so $(x_1 * x_3) * x_2 = 0$

Sufficient conditions for associative law

1. At least two of x_1, x_2 and x_3 have compact supports[3]
2. x_1, x_2, x_3 all right-sided (or left-sided), $\implies x_1 * x_2 * x_3$ also right-sided (or left-sided)
3. One signal (say x_3) has finite L_p norm for $1 \leq p \leq \infty$ and others finite L_1 norm.

$$\|x_1 * x_2 * x_3\|_p \leq \|x_1\|_1 \cdot \|x_2\|_1 \cdot \|x_3\|_p$$

[3] δ_a and its derivatives (to be defined) have support $\{a\}$.

Contents

Memory

For LTI systems

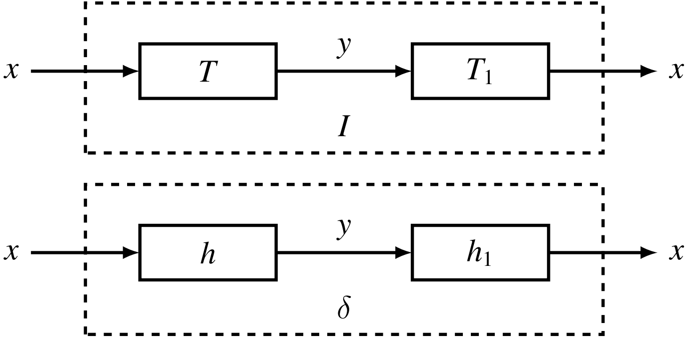
$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k], \quad \forall n \in \mathbb{Z}$$

$$y(t) = (x * h)(t) = \int_{\mathbb{R}} x(\tau)h(t-\tau)d\tau, \quad \forall t \in \mathbb{R}$$

memroyless $\iff h = K\delta$

All LTI systems except for scalar multiplication have memory

Invertibility



Impulse responses of a system and its inverse satisfy

$$h * h_1 = \delta$$

Necessary but **not** sufficient (requires associativity)

- e.g. first difference $h = \delta - \tau_1\delta$, accumulator $h_1 = u$

Causality

For LTI systems

$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

causal $\iff h[n] = 0$ **for all** $n < 0$

$$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

causal $\iff h(t) = 0$ **for all** $t < 0$

Stability

Recall BIBO stability: $\|x\|_\infty < \infty \implies \|T(x)\|_\infty < \infty$

For LTI systems

$$\textbf{BIBO stable} \iff \|h\|_1 < \infty$$

Proof. Sufficiency. Assume $\|h\|_1 < \infty$. Recall $\|x * h\|_\infty \leq \|x\|_\infty \|h\|_1$. Thus $\|x\|_\infty < \infty \implies \|x * h\|_\infty < \infty$.

Necessity. Assume BIBO stability. Let $x = R(\bar{h}/|h|)$, where R is time reversal and $\bar{h}$ is complex conjugate of h [4]. Note $\|x\|_\infty = 1$. By stability, $\|x * h\|_\infty < \infty$. Note $\|h\|_1$ is value of $x * h$ at time zero. Thus $\|h\|_1 \leq \|x * h\|_\infty < \infty$.

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[4] when h takes zero value, use convention $0/0 = 0$.

Unit Step Response

Unit step response of LTI systems

$$s \triangleq T(u) = u * h$$

DT LTI

$$s[n] = \sum_{-\infty}^{n} h[k]$$

$$h[n] = s[n] - s[n-1]$$

CT LTI

$$s(t) = \int_{-\infty}^{t} h(\tau) d\tau$$

$$h(t) = s'(t)$$

Contents

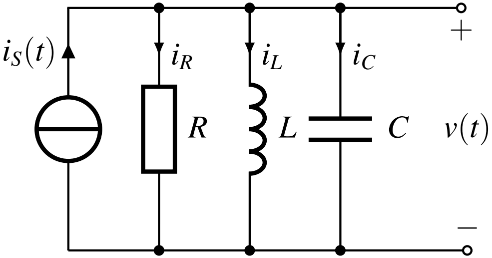
Linear Constant-coefficient Differential Equations

Characteristics of R, L, C

$$i_R(t) = \frac{1}{R}v(t)$$

$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau$$

$$i_C(t) = C\frac{d}{dt}v(t)$$



Kirchhoff's current law

$$i_R(t) + i_L(t) + i_C(t) = i_S(t)$$

Second order ordinary differential equation (ODE)

$$C\frac{d^2}{dt^2}v(t) + \frac{1}{R}\frac{d}{dt}v(t) + \frac{1}{L}v(t) = \frac{d}{dt}i_S(t)$$

Linear Constant-coefficient Differential Equations

System described by linear constant-coefficient ODE

$$L_y y = L_x x$$

where

$$L_y = \sum_{k=0}^{N} a_k \frac{d^k}{dt^k} \quad (a_N \neq 0), \quad L_x = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} \quad (b_M \neq 0)$$

- N: order of ODE
- input-output relation specified implicitly by ODE
- solve ODE for explicit input-output relation $y = T(x)$
- can take $f = L_x x$ as "input" when solving ODE
- ODE alone does **not** uniquely determine T
- need auxiliary conditions, typically initial conditions

Linear Constant-coefficient Differential Equations

Initial value problem (IVP)

$$L_y y = f$$

with initial conditions

$$y^{(k)}(t_0) = y_k, \quad k = 0, 1, \ldots, N-1$$

- N-th order ODE needs N initial conditions
- Replace y and f by $\tilde{y} = \tau_{-t_0} y$ and $\tilde{f} = \tau_{-t_0} f$,

  $$L_y \tilde{y} = \tilde{f}$$

  with initial conditions

  $$\tilde{y}^{(k)}(0) = y_k, \quad k = 0, 1, \ldots, N-1$$

IVP with First-order ODE

Example. $L_y = \frac{d}{dt} + 2$, i.e.

$$y'(t) + 2y(t) = x(t) \qquad (1)$$

with input $x(t) = Ke^{3t}u(t)$ and initial condition $y(0) = y_0$.

- General solution is sum of particular solution $y_p(t)$ and homogeneous solution $y_h(t)$, i.e.

$$y(t) = y_p(t) + y_h(t)$$

- y_p satisfies (1); y_h (natural response) satisfies

$$y_h'(t) + 2y_h(t) = 0$$

- $y_h(t) = Ae^{\lambda t}$, where $\lambda + 2 = 0$; LHS obtained from L_y upon replacing $\frac{d}{dt}$ by λ. Thus $y_h(t) = Ae^{-2t}$.

IVP with First-order ODE

Example (cont'd). $L_y = \frac{d}{dt} + 2$, i.e.

$$y'(t) + 2y(t) = x(t) \tag{1}$$

with input $x(t) = Ke^{3t}u(t)$ and initial condition $y(0) = y_0$.

- For particular solution y_p, look for forced response, i.e. signal of same of as input.
- For $t > 0$, $x(t) = Ke^{3t}$, so assume $y_p(t) = Ye^{3t}$.

$$L_y y_p(t) = 5Ye^{3t} = x(t) = Ke^{3t} \implies y_p(t) = \frac{K}{5}e^{3t}$$

- General solution

$$y(t) = \frac{K}{5}e^{3t} + Ae^{-2t}, \quad t > 0$$

IVP with First-order ODE

Example (cont'd). $L_y = \frac{d}{dt} + 2$, i.e.

$$y'(t) + 2y(t) = x(t) \tag{1}$$

with input $x(t) = Ke^{3t}u(t)$ and initial condition $y(0) = y_0$.

- Use initial condition to determine A

$$y(0) = \frac{K}{5} + A = y_0 \implies A = y_0 - \frac{K}{5}$$

- Complete solution to IVP

$$y(t) = \underbrace{\frac{K}{5}e^{3t}}_{\text{forced response}} + \underbrace{\left(y_0 - \frac{K}{5}\right)e^{-2t}}_{\text{natural response}}, \quad t > 0$$

- $y(t) = y_0 e^{-2t}$ for $t \leq 0$, but typically interested in $t > 0$

IVP with First-order ODE

Example (cont'd). $L_y = \frac{d}{dt} + 2$, i.e.

$$y'(t) + 2y(t) = x(t) \tag{1}$$

with input $x(t) = Ke^{3t}u(t)$ and initial condition $y(0) = y_0$.

- Complete solution to IVP

$$y(t) = \frac{K}{5}e^{3t} + \left(y_0 - \frac{K}{5}\right)e^{-2t}, \quad t > 0$$

- Is the system $y = T(x)$ linear? **No** in general.
  - homogeneity fails if $y_0 \neq 0$, y not proportional to K.
- Rewrite solution as

$$y(t) = \underbrace{\frac{K}{5}(e^{3t} - e^{-2t})}_{\text{zero-state response}} + \underbrace{y_0 e^{-2t}}_{\text{zero-input response}}, \quad t > 0$$

Decomposition of Solutions (1)

Consider general linear ODE

$$L_y y = f$$

Particular solution

$$L_y y_p = f$$

Homogeneous solution

$$L_y y_h = 0$$

- Defined with respect to ODE alone!
- Nothing to do with initial conditions

Decomposition of Solutions (2)

Consider IVP with general linear ODE

$$\begin{cases} L_y y = f \\ y^{(k)}(0) = y_k, \quad k = 0, 1, \ldots, N-1 \end{cases} \tag{2}$$

Zero-input response

$$\begin{cases} L_y y_{zi} = 0 \\ y_{zi}^{(k)}(0) = y_k, \quad k = 0, 1, \ldots, N-1 \end{cases}$$

Zero-state response

$$\begin{cases} L_y y_{zs} = f \\ y_{zs}^{(k)}(0) = 0, \quad k = 0, 1, \ldots, N-1 \end{cases}$$

Complete solution to (2) is $y = y_{zi} + y_{zs}$.

Linearity

Zero-state response linear in input

$$\begin{cases} L_y y_{zs,i} = f_i \\ y_{zs,i}^{(k)}(0) = 0 \end{cases} \implies \begin{cases} L_y(\sum_i c_i y_{zs,i}) = \sum_i c_i f_i \\ (\sum_i c_i y_{zs,i})^{(k)}(0) = 0 \end{cases}$$

Zero-input response linear in initial state

$$\begin{cases} L_y y_{zs,i} = 0 \\ y_{zs,i}^{(k)}(0) = y_{k,i} \end{cases} \implies \begin{cases} L_y(\sum_i c_i y_{zs,i}) = 0 \\ (\sum_i c_i y_{zs,i})^{(k)}(0) = \sum_i c_i y_{k,i} \end{cases}$$

- Complete response is linear in input iff zero-input response is zero
- Linearity requires zero initial state

Time-invariance

Zero initial state, i.e. $y^{(k)}(0) = 0$ for $k = 0, \ldots, N-1$ guarantees linearity but not time-invariance

Example. Consider

$$y'(t) + 2y(t) = x(t)$$

with initial condition $y(0) = 0$.

- If $x(t) = u(t)$,

$$y(t) = \frac{1}{2}(1 - e^{-2t})u(t)$$

- If $x(t) = u(t+1)$,

$$y(t) = \frac{1}{2}(1 - e^{-2t})u(t+1) + \frac{1}{2}(e^{-2} - 1)e^{-2t}u(-t-1)$$

Initial Rest

Often work with right-sided inputs, i.e. $x(t) = 0$ for $t < t_0$

- stimulus turned on at some point

Initial rest condition

- If input $x(t) = 0$ for $t < t_0$, output $y(t) = 0$ for $t < t_0$
  - equivalent to causality for linear systems

- Use initial condition $y^{(k)}(t_0) = 0$ for $k = 0, 1, \ldots, N-1$, i.e. solve

$$\begin{cases} L_y y = f \\ y^{(k)}(t_0) = 0, \quad k = 0, 1, \ldots, N-1 \end{cases}$$

Linear constant-coefficient ODE with initial rest condition specifies causal and LTI system **for right-sided inputs**