EI331 Signals and Systems

Lecture 7

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Initial Rest

Often work with right-sided inputs, i.e. $x(t) = 0$ for $t < t_0$

- stimulus turned on at some point

Initial rest condition

- If input $x(t) = 0$ for $t < t_0$, output $y(t) = 0$ for $t < t_0$
  - output zero until changed by input (cf. Newton's law)
  - equivalent to causality for linear systems
- Adapt initial time t_0 to input x: if x becomes nonzero at t_0, use $y^{(k)}(t_0) = 0$ for $k = 0, 1, \ldots, N-1$, i.e. solve

$$\begin{cases} L_y y = f \\ y^{(k)}(t_0) = 0, \quad k = 0, 1, \ldots, N-1 \end{cases}$$

Linear constant-coefficient ODE with initial rest condition specifies causal and LTI system **for right-sided inputs**

Initial Rest

Example. Newton's second law

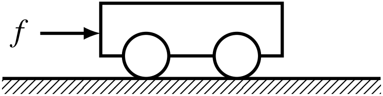
$$mx''(t) = f(t)$$

Initial rest

- stays at origin $x = 0$
- zero velocity $v = 0$ (at rest!)
- stays so unless changed by external force (Newton's first law)

If force starts at $t = 0$

- $x(0) = 0, v(0) = x'(0) = 0$



If force starts on at $t = 1$

- $x(1) = 0, v(1) = x'(1) = 0$

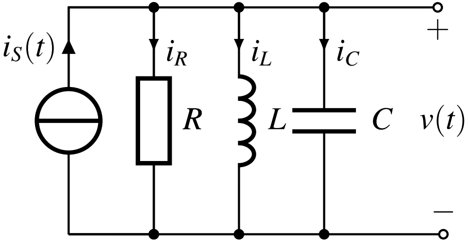
Initial Rest

Example. RLC circuit

$$C\frac{d^2}{dt^2}v(t) + \frac{1}{R}\frac{d}{dt}v(t) + \frac{1}{L}v(t) = \frac{d}{dt}i_S(t)$$

Initial rest

- no stored energy in L, C
- zero voltage and current



If source on at $t = 0$

- $v(0) = 0$
- $i_C(0) = Cv'(0) = 0$

If source on at $t = 1$

- $v(1) = 0$
- $i_C(1) = Cv'(1) = 0$

Initial Rest

General IVP with first-order ODE

$$\begin{cases} y'(t) + ay(t) = f(t) \\ y(t_0) = y_0 \end{cases}$$

Solution

$$y(t) = \underbrace{y_0 e^{-a(t-t_0)}}_{\text{zero-input response}} + \underbrace{\int_{t_0}^{t} f(\tau) e^{-a(t-\tau)} d\tau}_{\text{zero-state response}}$$

Initial rest: zero-input response **always** 0; take $t_0 \to -\infty$

$$y(t) = \int_{-\infty}^{t} f(\tau) e^{-a(t-\tau)} d\tau \xRightarrow{f = f \cdot \tau_{t_0} u} y(t) = u(t-t_0) \int_{t_0}^{t} f(\tau) e^{-a(t-\tau)} d\tau$$

Initial Rest

Example.

$$y'(t) + 2y(t) = x(t)$$

with initial rest condition and input $x(t) = u(t+1)$

Response

$$y(t) = \int_{-\infty}^{t} x(\tau)e^{-2(t-\tau)}d\tau = \int_{-\infty}^{t} u(\tau+1)e^{-2(t-\tau)}d\tau$$

For $t < -1$

$$y(t) = \int_{-\infty}^{t} 0 \cdot e^{-2(t-\tau)}d\tau = 0$$

For $t > -1$

$$y(t) = \int_{-1}^{t} e^{-2(t-\tau)}d\tau = \frac{1}{2}(1 - e^{-2(t+1)})$$

Non-initial Rest

Example.

$$y'(t) + 2y(t) = x(t)$$

with initial condition $y(0) = 0$ and input $x(t) = u(t+1)$

Response

$$y(t) = \int_0^t x(\tau)e^{-2(t-\tau)}d\tau = \int_0^t u(\tau+1)e^{-2(t-\tau)}d\tau$$

For $t > -1$

$$y(t) = \int_0^t e^{-2(t-\tau)}d\tau = \frac{1}{2}(1 - e^{-2t})$$

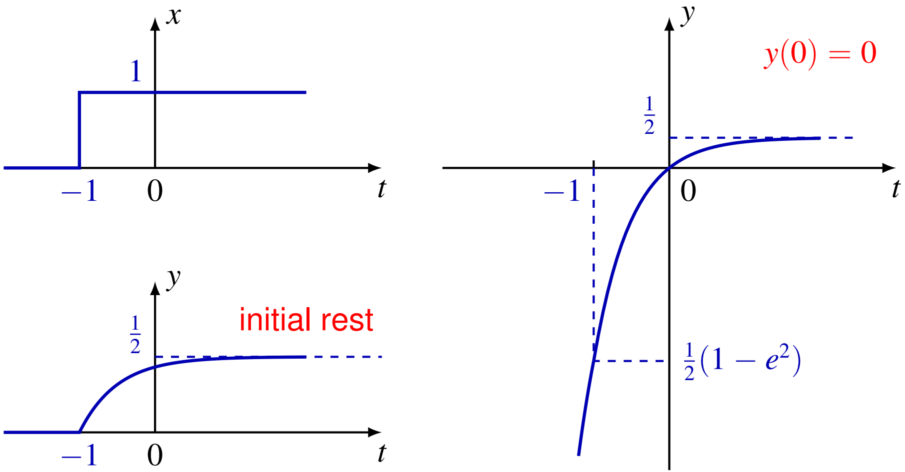
For $t < -1$

$$y(t) = \int_0^{-1} e^{-2(t-\tau)}d\tau = \frac{1}{2}(e^{-2} - 1)e^{-2t}$$

Comparison of Initial Conditions

Example.

$$y'(t) + 2y(t) = u(t+1)$$



Jump from 0_- to 0_+

Need more care for initial condition with singular input

Example. What's impulse response of causal LTI system described by $y'(t) + 2y(t) = x(t)$ with initial rest condition?

Method 1. Solve

$$\begin{cases} h'(t) + 2h(t) = \delta(t) \\ h(0) = 0 \end{cases}$$

- For $t \neq 0$, reduces to

$$\begin{cases} h'(t) + 2h(t) = 0 \\ h(0) = 0 \end{cases}$$

- General solution $h(t) = Ae^{-2t}$ for $t \neq 0$
- $h(0) = 0 \implies h(t) = 0$ for $t \neq 0$, **something wrong** $\lightning$

Jump from 0_- to 0_+

Need more care for initial condition with singular input

Example. What's impulse response of causal LTI system described by $y'(t) + 2y(t) = x(t)$ with initial rest condition?

Method 2. Response

$$y(t) = \int_{-\infty}^{t} x(\tau)e^{-2(t-\tau)}d\tau \implies h(t) = e^{-2t}u(t)$$

Observation: h discontinuous at $t = 0$

- $h(0_-) = 0$, $h(0_+) = 1$, due to singularity of δ at $t = 0$
- For $t > 0$,

$$h(t) = \int_{-\infty}^{t} \delta(\tau)e^{-2(t-\tau)}d\tau = \int_{0_-}^{t} \delta(\tau)e^{-2(t-\tau)}d\tau$$

Jump from 0_- to 0_+

IVP

$$\begin{cases} y'(t) + ay(t) = f(t) \\ y(0_-) = y_0 \end{cases} \quad \text{vs.} \quad \begin{cases} y'(t) + ay(t) = f(t) \\ y(0_+) = y_0 \end{cases}$$

Solution

$$y(t) = y(0_-)e^{-at} + \int_{0_-}^{t} f(\tau)e^{-a(t-\tau)}d\tau$$

vs.

$$y(t) = y(0_+)e^{-at} + \int_{0_+}^{t} f(\tau)e^{-a(t-\tau)}d\tau$$

Initial rest: use $y(0_-) = 0$

$$y(0_+) = y(0_-) + \int_{0_-}^{0_+} f(\tau)e^{a\tau}d\tau$$

- if $f(\tau)$ has no singularity at $\tau = 0$, $y(0_+) = y(0_-) = 0$
- if $f(\tau)$ has singularity at $\tau = 0$, $y(0_+)$ may be different

Jump from 0_- to 0_+

Example. Impulse response of $y'(t) + 2y(t) = x(t)$ revisited.

$$\begin{cases} h'(t) + 2h(t) = \delta(t) \\ h(0_-) = 0 \end{cases}$$

- For $t \neq 0$, reduces to

$$\begin{cases} h'(t) + 2h(t) = 0, \quad t > 0 \\ h(0_-) = 0 \end{cases} \qquad \begin{cases} h'(t) + 2h(t) = 0, \quad t < 0 \\ h(0_-) = 0 \end{cases}$$

- General solution $h(t) = A_+ e^{-2t} u(t) + A_- e^{-2t} u(-t)$
- $A_+ = h(0_+)$, but used $A_+ = A_- = h(0_-) = 0$ in first try ↯

$$\xcancel{h(t) = h(0_-)e^{-at} + \int_{0_+}^{t} \delta(\tau) e^{-a(t-\tau)} d\tau}$$

Recipe for IVP with First-order ODE

IVP

$$\begin{cases} y'(t) + ay(t) = f(t) \\ y(t_0) = y_0 \end{cases}$$

Solution for **all cases**

$$\boxed{y(t) = y(t_0)e^{-a(t-t_0)} + \int_{t_0}^{t} f(\tau)e^{-a(t-\tau)}d\tau}$$

- If t_0 means t_{0+} or t_{0-}, be **consistent** in all places!
- Matters only if f has singularity at t_0

Initial rest

$$\boxed{y(t) = \int_{-\infty}^{t} f(\tau)e^{-a(t-\tau)}d\tau}$$

Higher-order ODE

$$Ly = \sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y = f, \quad (a_N \neq 0)$$

General solution

$$y = \underbrace{y_h}_{\text{homogeneous solution}} + \underbrace{y_p}_{\text{particular solution}}$$

Characteristic equation

$$\sum_{k=0}^{N} a_k \lambda^k = 0$$

- LHS obtained from L by substituion $\frac{d}{dt} \to \lambda$; note $\frac{d^k}{dt^k} = \left(\frac{d}{dt}\right)^k$
- N (complex) roots by Fundamental Theorem of Algebra (root of multiplicity k counted as k roots)

Higher-order ODE

Homogeneous solution

- r distinct characteristic roots λ_i of multiplicity m_i, $i = 1, 2, \ldots, r$ (note $\sum_{i=1}^{r} m_i = N$)
- Homogeneous solution takes form

$$y_h(t) = \sum_{i=1}^{r} \sum_{k=1}^{m_i} A_{ik} t^{k-1} e^{\lambda_i t}$$

  i.e. space of all homogeneous solutions has basis

$$e^{\lambda_1 t}, te^{\lambda_1 t}, \ldots, t^{m_1 - 1} e^{\lambda_1 t}; \ldots; e^{\lambda_r t}, te^{\lambda_r t}, \ldots, t^{m_r - 1} e^{\lambda_r t}.$$

- When $a_k \in \mathbb{R}$, $\forall k$, complex roots $\sigma \pm j\omega$ appear in pairs
  - in calculus, used $e^{\sigma t} \cos(\omega t)$ and $e^{\sigma t} \sin(\omega t)$
  - here, use $e^{(\sigma + j\omega)t}$ and $e^{(\sigma - j\omega)t}$
  - equivalent by Euler's formula

Higher-order ODE

Particular solution

- Look for forced response of same form as input f

f	y_p
t^p, 0 not characteristic root	$\sum_{k=0}^{p} B_k t^k$
t^p, 0 characteristic root of multiplicity m	$\sum_{k=0}^{p} B_k t^{m+k}$
e^{at}, a not characteristic root	Be^{at}
$e^{\lambda_i t}$, λ_i characteristic root of multiplicity m_i	$Bt^{m_i}e^{\lambda_i t}$

Note $\cos(\omega t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$ and $\sin(\omega t) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$ are special cases

IVP with Second-order ODE

Example. Second-order system

$$y'' + 3y' + 2y = x$$

at initial rest. Let $x(t) = e^{-t}u(t)$.

- Characteristic equation

$$\lambda^2 + 3\lambda + 2 = 0 \implies \lambda_1 = -1, \lambda_2 = -2$$

- Homogeneous solution $y_h(t) = A_1e^{-t} + A_2e^{-2t}$
- For $t > 0$, particular solution $y_p(t) = Bte^{-t}$

$$y_p''(t) + 3y_p'(t) + 2y_p(t) = Be^{-t} = x(t) = e^{-t} \implies B = 1$$

- General solution $y(t) = te^{-t} + A_1e^{-t} + A_2e^{-2t}$
- Initial rest $y(0) = y'(0) = 0 \implies y(t) = te^{-t} + e^{-2t} - e^{-t}$

Systems of First-order ODEs

Consider N-th order ODE with $a_N = 1$ (WLOG)

$$y^{(N)} + a_{N-1}y^{(N-1)} + \cdots + a_1 y' + a_0 y = f \qquad (1)$$

Let $Y_k = y^{(k)}$, $k = 0, 1, \ldots, N-1$

- $Y_k' = Y_{k+1}$ for $k = 0, 1, \ldots, N-2$
- $Y_{N-1}' = y^{(N)} = f - \sum_{k=0}^{N-1} a_k y^{(k)} = f - \sum_{k=0}^{N-1} a_k Y_k$

(1) equivalent to

$$Y' = AY + bf$$

where

$$Y = \begin{pmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_{N-2} \\ Y_{N-1} \end{pmatrix}, A = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \ldots & -a_{N-1} \end{pmatrix}, b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Systems of First-order ODEs

Initial value problem (IVP)

$$\begin{cases} y^{(N)} + a_{N-1}y^{(N-1)} + \cdots + a_1 y' + a_0 y = f \\ y^{(k)}(t_0) = y_k, \quad k = 0, 1, \ldots, N-1 \end{cases} \tag{2}$$

equivalent to

$$\begin{cases} Y' = AY + bf \\ Y(t_0) = Y_0 \end{cases} \tag{3}$$

where $Y_0 = (y_0, y_1, \ldots, y_{N-1})^T$.

Solution to (3)

$$Y(t) = \underbrace{e^{A(t-t_0)}Y_0}_{\text{zero-input response}} + \underbrace{\int_{t_0}^{t} f(\tau)e^{A(t-\tau)}bd\tau}_{\text{zero-state response}}$$

matrix exponential $e^{At} \triangleq \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = I + At + \frac{1}{2}(At)^2 + \ldots$

Systems of First-order ODEs

Solution to (2)

$$y(t) = cY(t) = \underbrace{ce^{A(t-t_0)}Y_0}_{\text{zero-input response}} + \underbrace{\int_{t_0}^{t} f(\tau)ce^{A(t-\tau)}bd\tau}_{\text{zero-state response}}$$

where $c = (1, 0, 0, \ldots, 0)$

Initial rest

$$y(t) = \int_{-\infty}^{t} f(\tau)ce^{A(t-\tau)}bd\tau$$

- Recall $f = \sum_{k=0}^{M} b_k x^{(k)}$ linear in x
- $y = T(x)$ causal LTI system; if $f = x$, $h(t) = ce^{At}bu(t)$

IVP with Second-order ODE Revisited

Example. Let $x(t) = e^{-t}u(t)$. Consider IVP

$$\begin{cases} y'' + 3y' + 2y = x \\ y(0_-) = y_0, y'(0_-) = y_1 \end{cases}$$

- Let $Y = (y, y')^T$.

$$Y' = AY + bx$$

  where

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- For $t > 0$,

$$y(t) = (1, 0)e^{At} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} + \int_{0_-}^{t} x(\tau)(1, 0)e^{A(t-\tau)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} d\tau$$

- Need to compute e^{At}

IVP with Second-order ODE Revisited

Example (cont'd). Let $x(t) = e^{-t}u(t)$. Consider IVP

$$\begin{cases} y'' + 3y' + 2y = x \\ y(0_-) = y_0, y'(0_-) = y_1 \end{cases}$$

- Diagonalize[1] A

$$A = P\Lambda P^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

- Exponentiate At

$$e^{At} = \sum_{n=0}^{\infty} P\frac{(\Lambda t)^n}{n!}P^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

[1] Not always possible; may need Jordan canonical form in general.

IVP with Second-order ODE Revisited

Example (cont'd). Let $x(t) = e^{-t}u(t)$. Consider IVP

$$\begin{cases} y'' + 3y' + 2y = x \\ y(0_-) = y_0, y'(0_-) = y_1 \end{cases}$$

- Complete response

$$y(t) = (2y_0 + y_1)e^{-t} - (y_0 + y_1)e^{-2t} + \int_{0_-}^{t} x(\tau)g(t-\tau)d\tau$$

where

$$g(t) = (1,0)e^{At}\begin{pmatrix}0\\1\end{pmatrix} = e^{-t} - e^{-2t}$$

- Zero-state response

$$y_{zs}(t) = \int_0^t e^{-\tau}[e^{-(t-\tau)} - e^{-2(t-\tau)}]d\tau = te^{-t} + e^{-2t} - e^{-t}$$

IVP with Second-order ODE Revisited

Example. Consider second-order system at initial rest

$$y'' + 3y' + 2y = x$$

- Response

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = (x * h)(t)$$

  where

$$h(t) = (e^{-t} - e^{-2t})u(t)$$

- If $x(t) = e^{-t}u(t)$,

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} e^{-\tau}u(\tau)[e^{-(t-\tau)} - e^{-2(t-\tau)}]d\tau \\ &= (te^{-t} + e^{-2t} - e^{-t})u(t) \end{aligned}$$

Duhamel's Principle

Solution to vector IVP with first-order ODE

$$Y(t) = \underbrace{e^{A(t-t_0)}Y(t_0)}_{\text{zero-input response}} + \underbrace{\int_{t_0}^{t} f(\tau)G(t-\tau)d\tau}_{\text{zero-state response}}$$

- $G(t) = e^{At}b$ is homogeneous solution to $G' = AG$ with initial condition $G(0) = b = (0, \ldots, 0, 1)^T$

Solution to scalar IVP with higher-order ODE

$$y(t) = \underbrace{ce^{A(t-t_0)}Y(t_0)}_{\text{zero-input response}} + \underbrace{\int_{t_0}^{t} f(\tau)g(t-\tau)d\tau}_{\text{zero-state response}}$$

- $g(t) = ce^{At}b = cG(t)$ is homogeneous solution to $Lg = 0$ with initial condition $G(0) = b$; recall $G = (g, g', g'', \ldots, g^{(N-1)})^T$.

Jean-Marie Duhamel (from Wikipedia)

Impulse Response of Higher-order ODE

Impulse response of 1D system at initial rest

$$\begin{cases} a_N h^{(N)} + a_{N-1} h^{(N-1)} + \cdots + a_1 h' + a_0 h = \delta \\ h^{(k)}(0_-) = 0, \quad k = 0, 1, \ldots, N-1 \end{cases}$$

By Duhamel's principle

$$h(t) = ce^{At} h(0_-) + \int_{0_-}^{t} \delta(\tau) g(t-\tau) d\tau = g(t)u(t)$$

where g satisfies

$$\begin{cases} a_N g^{(N)} + a_{N-1} g^{(N-1)} + \cdots + a_1 g' + a_0 g = 0 \\ g^{(k)}(0) = 0,\ k = 0, 1, \ldots, N-2; \quad g^{(N-1)}(0) = 1/a_N \end{cases}$$

Second Recipe for IVP with Higher-order ODE

IVP

$$\begin{cases} \sum_{k=0}^{N} a_k y^{(k)}(t) = f(t) \\ y^{(k)}(t_0) = y_k, \quad k = 0, 1, \ldots, N-1 \end{cases}$$

1. Find homogeneous solution y_h
2. Find zero-input response y_{zi}, i.e. homogeneous solution satisfying initial condition $y_{zi}^{(k)}(t_0) = y_k$, $k = 0, 1, \ldots, N-1$
3. Find homogeneous solution g satisfying initial condition $g^{(k)}(0) = 0, 0 \leq k \leq N-2$ and $g^{(N-1)}(0) = 1/a_N$
4. Find zero-state response $y_{zs}(t) = \int_{t_0}^{t} f(\tau)g(t-\tau)d\tau$
5. Complete solution $y = y_{zi} + y_{zs}$

Second Recipe for IVP with Higher-order ODE

Example. Let $x(t) = e^{-t}u(t)$. Consider IVP

$$\begin{cases} y'' + 3y' + 2y = x \\ y(0_-) = y_0, y'(0_-) = y_1 \end{cases}$$

- Homogeneous solution $y_h(t) = A_1 e^{-t} + A_2 e^{-2t}$
- Zero-input response

$$\begin{cases} y(0_-) = A_1 + A_2 = y_0 \\ y'(0_-) = -A_1 - 2A_2 = y_1 \end{cases} \implies \begin{cases} A_1 = 2y_0 + y_1 \\ A_2 = -(y_0 + y_1) \end{cases}$$

so

$$y_{zi}(t) = (2y_0 + y_1)e^{-t} - (y_0 + y_1)e^{-2t}$$

Second Recipe for IVP with Higher-order ODE

Example (cont'd). Let $x(t) = e^{-t}u(t)$. Consider IVP

$$\begin{cases} y'' + 3y' + 2y = x \\ y(0_-) = y_0, y'(0_-) = y_1 \end{cases}$$

- Homogeneous solution $y_h(t) = A_1 e^{-t} + A_2 e^{-2t}$
- Homogeneous solution g

$$\begin{cases} g(0) = A_1 + A_2 = 0 \\ g'(0) = -A_1 - 2A_2 = 1 \end{cases} \implies \begin{cases} A_1 = 1 \\ A_2 = -1 \end{cases}$$

so

$$g(t) = e^{-t} - e^{-2t}$$

- Zero-state response

$$y_{zs}(t) = \int_0^t e^{-\tau}[e^{-(t-\tau)} - e^{-2(t-\tau)}]d\tau = te^{-t} + e^{-2t} - e^{-t}$$

Second Recipe for IVP with Higher-order ODE

Example. Find impulse response of following LTI system

$$y'' + 3y' + 2y = x$$

- By previous example,

$$h(t) = (e^{-t} - e^{-2t})u(t)$$

- Alternatively, can find step response

$$s'' + 3s' + 2s = u$$

$$s(t) = \left(\frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}\right)u(t)$$

Then use

$$h(t) = s'(t)$$

Second Recipe for IVP with Higher-order ODE

Example. Consider LTI system

$$y'' + 3y' + 2y = x' + x$$

Want to find impulse response, i.e. solve

$$h'' + 3h' + 2h = \delta' + \delta$$

By previous example, impulse response of following LTI system

$$y'' + 3y' + 2y = x$$

is

$$h_1(t) = (e^{-t} - e^{-2t})u(t)$$

Then

$$h = h_1 * (\delta' + \delta) = h_1 + h_1' \implies h(t) = e^{-2t}u(t)$$