EI331 Signals and Systems

Lecture 8

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Contents

Linear Constant-coefficient Difference Equations

Example. Balance of bank account

$$y[n] = (1+r)y[n-1] + x[n] \quad r \text{ interest rate}$$

Example. Discretization of differential equation

$$y'(t) = x(t) \implies \frac{y(nT) - y((n-1)T)}{T} \approx x(nT)$$

Let $x[n] = x(nT)$, $y[n] = y(nT)$. Discretized equation

$$y[n] = y[n-1] + Tx[n] \quad \text{(Euler's method)}$$

Example. Exponential smoothing

$$y[n] = (1-\alpha)y[n-1] + \alpha x[n], \quad \alpha \in (0,1) \text{ smoothing factor}$$

Linear Constant-coefficient Difference Equations

System described by linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=-M_1}^{M} b_k x[n-k]$$

where $a_0 \neq 0$, $a_N \neq 0$

- also called recursive equation or recursion
- N: order of difference equation
- focus on $M_1 = 0$ for causal systems
- input-output relation specified implicitly
- solve difference equation for explicit input-output relation
- difference equation alone does **not** uniquely determine T
- need auxiliary conditions, typically initial conditions

Linear Constant-coefficient Differential Equations

Initial value problem (IVP)

$$Ly = f, \quad \text{where } L = \sum_{k=0}^{N} a_k \tau_k$$

with initial conditions

$$y[k] = y_k, \quad k = n_0 - 1, n_0 - 2, \ldots, n_0 - N$$

- N-th order difference equation needs N initial conditions
- may use any N consecutive values as “initial” values
- often $n_0 = 0$
- typically $f[n] = 0$ for $n < n_0$

Initial Rest

- If input $x[n] = 0$ for $n < n_0$, output $y[n] = 0$ for $n < n_0$
  - output zero until changed by input
  - equivalent to causality for linear systems
- Adapt initial time n_0 to input x: if x becomes nonzero at n_0, use $y[n_0 - k] = 0$ for $k = 1, 2, \ldots, N$, i.e. solve

$$\begin{cases} Ly = f \\ y[k] = 0, \quad k = n_0 - 1, n_0 - 2, \ldots, n_0 - N \end{cases}$$

- Linear constant-coefficient difference equation with initial rest condition specifies causal and LTI system **for right-sided inputs**

Iterative Method

Iteratively compute $y[n]$ from $y[n-1], \ldots, y[n-N]$ and x

$$y[n] = \frac{1}{a_0}\left(\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right)$$

Special case $N = 0$

$$y[n] = \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) x[n-k]$$

- explicit function of present and past input values
- nonrecursive equation, no need for auxiliary conditions
- causal LTI system with finite impulse response (FIR)

$$h[n] = \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) \delta[n-k]$$

Iterative Method

Example. Consider exponential smoothing

$$y[n] - ay[n-1] = x[n]$$

with $y[-1] = y_{-1}$.

- For $n \geq 0$, go forward in time

$$\begin{aligned}
y[0] &= ay[-1] + x[0] = ay_{-1} + x[0] \\
y[1] &= ay[0] + x[1] = a^2 y_{-1} + ax[0] + x[1] \\
y[2] &= ay[1] + x[2] = a^3 y_{-1} + a^2 x[0] + ax[1] + x[2] \\
&\vdots \\
y[n] &= ay[n-1] + x[n] = a^{n+1} y_{-1} + \sum_{k=0}^{n} a^{n-k} x[k]
\end{aligned}$$

Iterative Method

Example (cont'd). Consider exponential smoothing

$$y[n] - ay[n-1] = x[n]$$

with $y[-1] = y_{-1}$.

- For $n \geq 0$, go forward in time

$$\begin{aligned} y[n] &= ay[n-1] + x[n] \\ ay[n-1] &= a^2 y[n-2] + ax[n-1] \\ &\vdots \\ a^{n-1}y[1] &= a^n y[0] + a^{n-1}x[1] \\ a^n y[0] &= a^{n+1}y[-1] + a^n x[0] \end{aligned}$$

Summing up all equations, $y[n] = a^{n+1}y_{-1} + \sum_{k=0}^{n} a^{n-k}x[k]$

Iterative Method

Example (cont'd). Consider exponential smoothing

$$y[n] - ay[n-1] = x[n]$$

with $y[-1] = y_{-1}$.

- $n < -1$, go backward in time

$$y[-2] = a^{-1}y[-1] - a^{-1}x[-1] = a^{-1}y_{-1} - a^{-1}x[-1]$$

$$y[-3] = a^{-1}y[-2] - a^{-1}x[-2] = a^{-2}y_{-1} - \sum_{k=-2}^{-1} a^{-3-k}x[k]$$

$$\vdots$$

$$y[n] = a^{-1}y[n+1] - a^{-1}x[n+1] = a^{n+1}y_{-1} - \sum_{k=n+1}^{-1} a^{n-k}x[k]$$

Iterative Method

Example (cont'd). Consider difference equation

$$y[n] - ay[n-1] = x[n]$$

with $y[-1] = y_{-1}$.

- Response

$$y[n] = \underbrace{a^{n+1} y_{-1}}_{\text{zero-input response}} + \underbrace{\left(\sum_{k=0}^{n} - \sum_{k=n+1}^{-1} \right) a^{n-k} x[k]}_{\text{zero-state response}}$$

  where by convention $\sum_{k=m_1}^{m_2} \cdot = 0$ if $m_2 < m_1$
- Initial rest: causal LTI system with impulse response

$$h[n] = a^n u[n]$$

  - infinite impulse response (IIR) system

Decomposition of Solutions (1)

Linear inhomogeneous difference equation

$$Ly = f, \quad \text{where } L = \sum_{k=0}^{N} a_k \tau_k$$

Particular solution

$$Ly_p = f$$

Homogeneous solution (natural response)

$$Ly_h = 0$$

General solution

$$y = y_p + y_h$$

Homogeneous Solution

Characteristic equation

$$\sum_{k=0}^{N} a_k \alpha^{N-k} = 0$$

- Obtained from $Ly_h = 0$ upon substitution of $y_h[n] = \alpha^n$
- r distinct characteristic roots α_i of multiplicity m_i, $i = 1, 2, \ldots, r$; note $\sum_{i=1}^{r} m_i = N$
- Homogeneous solution takes form

$$y_h[n] = \sum_{i=1}^{r} \sum_{k=1}^{m_i} A_{ik} n^{k-1} \alpha_i^n$$

i.e. space of all homogeneous solutions has basis

$$\alpha_1^n, n\alpha_1^n, \ldots, n^{m_1-1}\alpha_1^n; \ldots; \alpha_r^n, n\alpha_r^n, \ldots, n^{m_1-1}\alpha_r^n.$$

Particular Solution

- Look for forced response of same form as input f

f	y_p
n^p, 1 not characteristic root	$\sum_{k=0}^{p} B_k n^k$
n^p, 1 characteristic root of multiplicity m	$\sum_{k=0}^{p} B_k n^{m+k}$
α^n, α not characteristic root	$B\alpha^n$
α_i^n, α_i characteristic root of multiplicity m_i	$Bn^{m_i}\alpha_i^n$

Note $\cos(\omega n) = \frac{1}{2}\left(e^{j\omega n} + e^{-j\omega n}\right)$ and $\sin(\omega n) = \frac{1}{2j}\left(e^{j\omega n} - e^{-j\omega n}\right)$ are special cases

Example

Fibonacci sequence

$$\begin{cases} y[n] - y[n-1] - y[n-2] = 0 \\ y[1] = y[2] = 1 \end{cases}$$

- Characteristic equation

$$\alpha^2 - \alpha - 1 = 0 \implies \alpha_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

- Homogeneous solution

$$y[n] = A_1 \alpha_1^n + A_2 \alpha_2^n$$

- Determine A_1, A_2 from initial conditions

$$\begin{cases} y[1] = A_1 \alpha_1 + A_2 \alpha_2 = 1 \\ y[2] = A_1 \alpha_1^2 + A_2 \alpha_2^2 = 1 \end{cases} \implies \begin{cases} A_1 = \frac{1}{\sqrt{5}} \\ A_2 = -\frac{1}{\sqrt{5}} \end{cases}$$

Example

Solve difference equation

$$y[n] + 2y[n-1] = x[n] - x[n-1]$$

with input $x[n] = n^2$ and initial condition $y[-1] = -1$

- Characteristic equation $\alpha + 2 = 0 \implies \alpha = -2$
- Homogeneous solution $y_h[n] = A(-2)^n$
- Particular solution
  - find $f[n] = x[n] - x[n-1] = n^2 - (n-1)^2 = 2n - 1$
  - look for solution of form $y_p[n] = B_1 n + B_0$

$$y_p[n] + 2y_p[n-1] = 3B_1 n + 3B_0 - 2B_1 = f[n] = 2n - 1$$

  - compare coefficients

$$\begin{cases} 3B_1 = 2 \\ 3B_0 - 2B_1 = -1 \end{cases} \implies \begin{cases} B_1 = \frac{2}{3} \\ B_0 = \frac{1}{9} \end{cases}$$

Example (cont'd)

Solve difference equation

$$y[n] + 2y[n-1] = x[n] - x[n-1]$$

with input $x[n] = n^2$ and initial condition $y[-1] = -1$

- General solution

$$y[n] = \frac{2}{3}n + \frac{1}{9} + A(-2)^n$$

- Determine A from initial condition

$$y[-1] = \frac{2}{3}(-1) + \frac{1}{9} + A(-2)^{-1} = -1 \implies A = \frac{8}{9}$$

- Complete solution

$$y[n] = \underbrace{\frac{2}{3}n + \frac{1}{9}}_{\text{forced response}} + \underbrace{\frac{8}{9}(-2)^n}_{\text{natural response}}$$

Decomposition of Solutions (2)

IVP

$$\begin{cases} Ly = f \\ y[k] = y_k, \quad k = n_0 - 1, n_0 - 2, \ldots, n_0 - N \end{cases}$$

Zero-input response: linear in initial state

$$\begin{cases} Ly_{zi} = 0 \\ y_{zi}[k] = y_k, \quad k = n_0 - 1, n_0 - 2, \ldots, n_0 - N \end{cases}$$

Zero-state response: linear in input

$$\begin{cases} Ly_{zs} = f \\ y_{zs}[k] = 0, \quad k = n_0 - 1, n_0 - 2, \ldots, n_0 - N \end{cases}$$

Complete solution $\quad y = y_{zi} + y_{zs}$

Zero-state Response

$$\begin{cases} Ly_{zs} = f \\ y_{zs}[k] = 0, \quad k = n_0 - 1, n_0 - 2, \dots, n_0 - N \end{cases}$$

Focus on case where $f[n] = 0$ for $n < n_0 \implies$ initial rest

Causal and LTI system with output

$$y_{zs}[n] = (f * h)[n] = u[n - n_0] \sum_{k=n_0}^{n} f[k]h[n-k]$$

where h is impulse response of $Ly = x$, i.e.

$$\begin{cases} Lh = \delta, \quad n \geq 0 \\ h[k] = 0, \quad k = -1, -2, \dots, -N \end{cases}$$

or equivalently

$$\begin{cases} Lh = 0, \quad n \geq 1 \\ h[k] = 0, \quad k = -1, -2, \dots, 1 - N; h[0] = 1/a_0 \end{cases}$$

Example

Solve difference equation

$$y[n] + 2y[n-1] = x[n] - x[n-1]$$

with input $x[n] = n^2 u[n]$ and initial condition $y[-1] = -1$

- Homogeneous solution $y_h[n] = A(-2)^n$
- Zero-input response, i.e. homogeneous solution with $y_{zi}[-1] = -1$

$$A(-2)^{-1} = -1 \implies A = 2 \implies y_{zi} = 2(-2)^n$$

- Impulse response of $y[n] + 2y[n-1] = x[n]$, i.e. solution of following IVP

$$\begin{cases} h[n] + 2h[n-1] = \delta[n] \\ h[-1] = 0 \end{cases}$$

Example (cont'd)

Solve difference equation

$$y[n] + 2y[n-1] = x[n] - x[n-1]$$

with input $x[n] = n^2u[n]$ and initial condition $y[-1] = -1$

- Homogeneous solution $y_h[n] = A(-2)^n$
- Impulse response

$$\begin{cases} h[n] + 2h[n-1] = \delta[n], & n \geq 0 \\ h[-1] = 0 \end{cases}$$

  - by iterative method $h[0] = \delta[0] - 2h[-1] = 1$
  - now solve

$$\begin{cases} h[n] + 2h[n-1] = 0, & n \geq 1 \\ h[0] = 1 \end{cases}$$

  - $h[0] = A = 1 \implies h[n] = (-2)^n u[n]$

Example (cont'd)

Solve difference equation

$$y[n] + 2y[n-1] = x[n] - x[n-1]$$

with input $x[n] = n^2 u[n]$ and initial condition $y[-1] = -1$

- Impulse response $h[n] = (-2)^n u[n]$
- RHS $f[n] = x[n] - x[n-1] = (2n-1)u[n-1]$
- Zero-state response for $n \geq 1$

$$y_{zs}[n] = (f * h)[n] = \sum_{k=1}^{n} h[n-k](x[k] - x[k-1])$$

$$= \sum_{k=1}^{n} (-2)^{n-k}(2k-1) = \frac{2}{3}n + \frac{1}{9} - \frac{1}{9}(-2)^n$$

- Total response

$$y[n] = y_{zi}[n] + y_{zs}[n] = 2(-2)^n + \left[\frac{2}{3}n + \frac{1}{9} - \frac{1}{9}(-2)^n\right] u[n-1]$$

Systems of First-order Difference Equations

Consider N-th order difference equation with $a_0 = 1$

$$y + a_1\tau_1 y + \cdots + a_{N-1}\tau_{N-1}y + a_N\tau_N y = f$$

Let $Y_k = \tau_k y$, $k = 0, 1, \ldots, N-1$

- $Y_k = \tau_1 Y_{k-1}$ for $k = 1, 2, \ldots, N-1$
- $Y_0 = y = f - \sum_{k=1}^{N} a_k \tau_k y = f - \sum_{k=1}^{N} a_k \tau_1 Y_{k-1}$

Equivalent vector equation

$$Y = A\tau_1 Y + bf$$

where

$$Y = \begin{pmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_{N-2} \\ Y_{N-1} \end{pmatrix}, A = \begin{pmatrix} -a_1 & -a_2 & -a_3 & \ldots & -a_{N-1} & -a_N \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Systems of First-order Difference Equations

Initial value problem (IVP)

$$\begin{cases} y + a_1\tau_1 y + \cdots + a_{N-1}\tau_{N-1}y + a_N\tau_N y = f \\ y[k] = y_k, \quad k = n_0-1, n_0-2, \ldots, n_0-N \end{cases} \tag{1}$$

equivalent to

$$\begin{cases} Y = A\tau_1 Y + bf \\ Y[n_0-1] = Y_{n_0-1} \end{cases} \tag{2}$$

where $Y_{n_0-1} = (y_{n_0-1}, y_{n_0-2}, \ldots, y_{n_0-N})^T$.

Solution to (2)

$$Y[n] = \underbrace{A^{n-n_0+1}Y_{n_0-1}}_{\text{zero-input response}} + \underbrace{\left(\sum_{k=n_0}^{n} - \sum_{k=n+1}^{n_0-1}\right) A^{n-k} bf[k]}_{\text{zero-state response}}$$

Systems of First-order Difference Equations

Solution to (1)

$$y[n] = cY(t) = \underbrace{cA^{n-n_0+1}Y_{n_0-1}}_{\text{zero-input response}} + \underbrace{\left(\sum_{k=n_0}^{n} - \sum_{k=n+1}^{n_0-1}\right) f[k]cA^{n-k}b}_{\text{zero-state response}}$$

where $c = (1, 0, 0, \ldots, 0)$

Initial rest

$$y(t) = \sum_{k=-\infty}^{n} f[k]cA^{n-k}b$$

- Recall $f = \sum_{k=0}^{M} b_k \tau_k x$ linear in x
- $y = T(x)$ causal LTI system; if $f = x$, $h[n] = cA^n bu[n]$

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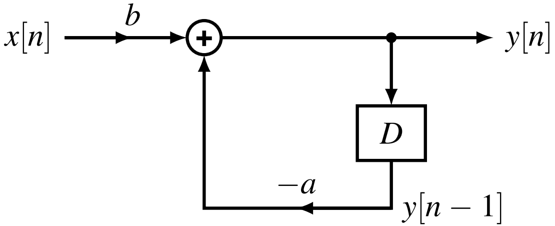
Block Diagram for $y[n] + ay[n-1] = bx[n]$

Rewrite input-output relation as

$$y[n] = -ay[n-1] + bx[n]$$



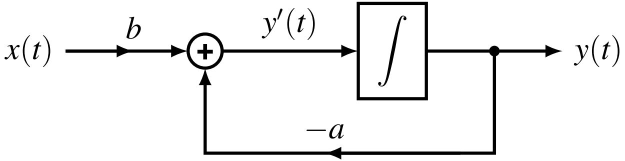
Basic elements

- adder
- scalar multiplication
- unit delay $D = \tau_1$

Block Diagram for $y'(t) + ay(t) = bx(t)$

Assuming $y(-\infty) = 0$, rewrite input-output relation as

$$y(t) = \int_{-\infty}^{t} [bx(\tau) - ay(\tau)]d\tau$$



Basic elements

- adder
- scalar multiplication
- integrator (preferred over differentiator for robustness)

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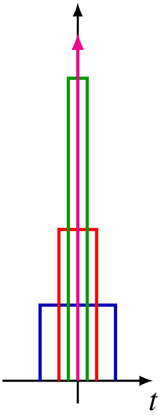
Unit Impulse Revisited

Recall unit impulse

$$\delta(t) = \lim_{\Delta \to 0} r_\Delta(t)$$

where

$$r_\Delta(t) = \frac{u(t + \frac{\Delta}{2}) - u(t - \frac{\Delta}{2})}{\Delta}$$



Idealization for quantities of very large magnitude but very small duration (e.g. impulse force) or spatial span (e.g. point mass/charge)

Recall unit impulse response

$$h = T(\delta) = \lim_{\Delta \to 0} T(r_\Delta)$$

Idealization: pulse so short that system response only depends on area, but not on shape and duration

Unit Impulse Revisited

For systems described by linear constant-coefficient ODE at initial rest

$$y(t) = \int_{\mathbb{R}} x(\tau)h(t-\tau)d\tau$$

Response to r_Δ

$$[T(r_\Delta)](t) = \int_{\mathbb{R}} r_\Delta(\tau)h(t-\tau)d\tau = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} h(t-\tau)d\tau$$

Indeed

$$\lim_{\Delta\to 0} T(r_\Delta) = h$$

at points of continuity of h. Same as

$$[T(\delta)](t) = \int_{\mathbb{R}} \delta(\tau)h(t-\tau)d\tau = h(t)$$

Unit Impulse Revisited

Recall second definition of δ

$$\int_{\mathbb{R}} \delta(t)\phi(t)dt = \phi(0)$$

Let $\mathcal{C}$ be set of functions continuous at 0. Above defines mapping

$$\begin{aligned} \delta : \mathcal{C} &\to \mathbb{R} \\ \phi &\mapsto \phi(0) \end{aligned}$$

Also denoted by $\delta[\phi] = \phi(0)$. δ is linear functional on $\mathcal{C}$

$$\delta[a_1\phi_1 + a_2\phi_2] = a_1\phi_1(0) + a_2\phi_2(0) = a_1\delta[\phi_1] + a_2\delta[\phi_2]$$

called generalized function or distribution

Unit Doublet

What's impulse response of differentiator?

$$h(t) = \delta'(t)$$

Expect output y of differentiator for input x to satisfy

$$y = x * h = x'$$

or

$$\int_{\mathbb{R}} x(\tau)\delta'(t - \tau)d\tau = x'(t)$$

at points where x is differentiable.

First definition of $u_1 = \delta'$

$$\boxed{x * \delta' = x'}$$

Unit Doublet

Second definition. Since $\delta \triangleq \lim_{\Delta \to 0} r_\Delta$, define

$$\delta' \triangleq \lim_{\Delta \to 0} r'_\Delta$$

meaning

$$\int_{\mathbb{R}} \delta'(t)\phi(t) = \lim_{\Delta \to 0} \int_{\mathbb{R}} r'_\Delta(t)\phi(t)dt$$

$$\begin{aligned}\int_{\mathbb{R}} r'_\Delta(t)\phi(t)dt &= \int_{\mathbb{R}} \frac{1}{\Delta}\left[\delta\left(t + \frac{\Delta}{2}\right) - \delta\left(t - \frac{\Delta}{2}\right)\right]\phi(t) \\ &= \frac{1}{\Delta}\left[\phi\left(-\frac{\Delta}{2}\right) - \phi\left(\frac{\Delta}{2}\right)\right]\end{aligned}$$

$$\boxed{\int_{\mathbb{R}} \delta'(t)\phi(t) = -\phi'(0)}$$

Unit Doublet

Third definition. For ϕ continuously differentiable at 0

$$\delta'[\phi] = -\delta[\phi']$$

Intuition: integration by parts should work

$$\int_{\mathbb{R}} f'(t)\phi(t)dt = [f(t)\phi(t)]\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} f(t)\phi'(t)dt$$

If ϕ has compact support, i.e. vanishes outside finite interval

$$\int_{\mathbb{R}} f'(t)\phi(t)dt = -\int_{\mathbb{R}} f(t)\phi'(t)dt$$

Take $f = \delta$

$$\boxed{\int_{\mathbb{R}} \delta'(t)\phi(t)dt = -\int_{\mathbb{R}} \delta(t)\phi'(t)dt}$$

Unit Doublet

Example. Show $f(t)\delta'(t) = f(0)\delta'(t) - f'(0)\delta(t)$

Proof.

$$\begin{aligned}
\int_{\mathbb{R}} f(t)\delta'(t)\phi(t)dt &= \int_{\mathbb{R}} \delta'(t)[f(t)\phi(t)]dt \\
&= -\int_{\mathbb{R}} \delta(t)[f(t)\phi(t)]'dt \\
&= -f'(0)\phi(0) - f(0)\phi'(0) \\
&= -\int_{\mathbb{R}} f'(0)\delta(t)\phi(t) + f(0)\int_{\mathbb{R}} \delta'(t)\phi(t)dt \\
&= \int_{\mathbb{R}} [f(0)\delta'(t) - f'(0)\delta(t)]\phi(t)dt
\end{aligned}$$

In particular, $t\delta'(t) = -\delta(t)$

Distributional Derivative

Distributional derivative of function(al) g defined by

$$\boxed{\int_{\mathbb{R}} g'(t)\phi(t)dt = -\int_{\mathbb{R}} g(t)\phi'(t)dt}$$

Example. Show $u'(t) = \delta(t)$

Proof. For continuously differentiable ϕ with compact support

$$\int_{\mathbb{R}} u'(t)\phi(t)dt = -\int_{\mathbb{R}} u(t)\phi'(t)dt = -\int_0^{\infty} \phi'(t)dt$$
$$= \phi(0) = \int_{\mathbb{R}} \delta(t)\phi(t)dt$$

Higher-order Derivatives of δ

First definition

$$\boxed{u_k = \delta^{(k)} = \underbrace{u_1 * u_1 * \cdots * u_1}_{k \text{ times}}, \quad k \geq 1}$$

Thus

$$u_k * f = f^{(k)}$$

Second definition

$$\boxed{\int_{\mathbb{R}} \delta^{(k)}(t)\phi(t)dt = (-1)^k \int_{\mathbb{R}} \delta(t)\phi^{(k)}(t)dt}$$

Use integration by parts k times

Higher-order Antiderivatives of δ

Let $u_0 = \delta$ be unit impulse, $u_{-1} = u$ unit step

$$\boxed{u_{-k} = \underbrace{u_{-1} * u_{-1} * \cdots * u_{-1}}_{k \textbf{ times}}, \quad k \geq 1}$$

Thus

$$u_{-k}(t) = \frac{t^{k-1}}{(k-1)!}u(t)$$

u_{-2} called unit ramp function

Property

$$\boxed{u_m * u_n = u_{m+n}, \quad m, n \in \mathbb{Z}}$$