Linear Programming

Linear Programming, LP Duality Theorem, LP-Relaxation

Linear Program (LP)

- A set of linear equations/inequalities.
- Maximize or minimize a given linear objective function.

$$
\begin{array}{ll}
\text{maximize} & c_1x_1 + c_2x_2 + \cdots c_nx_n \\
\text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \\
& x_1, x_2, \dots, x_n \ge 0
\end{array}
$$

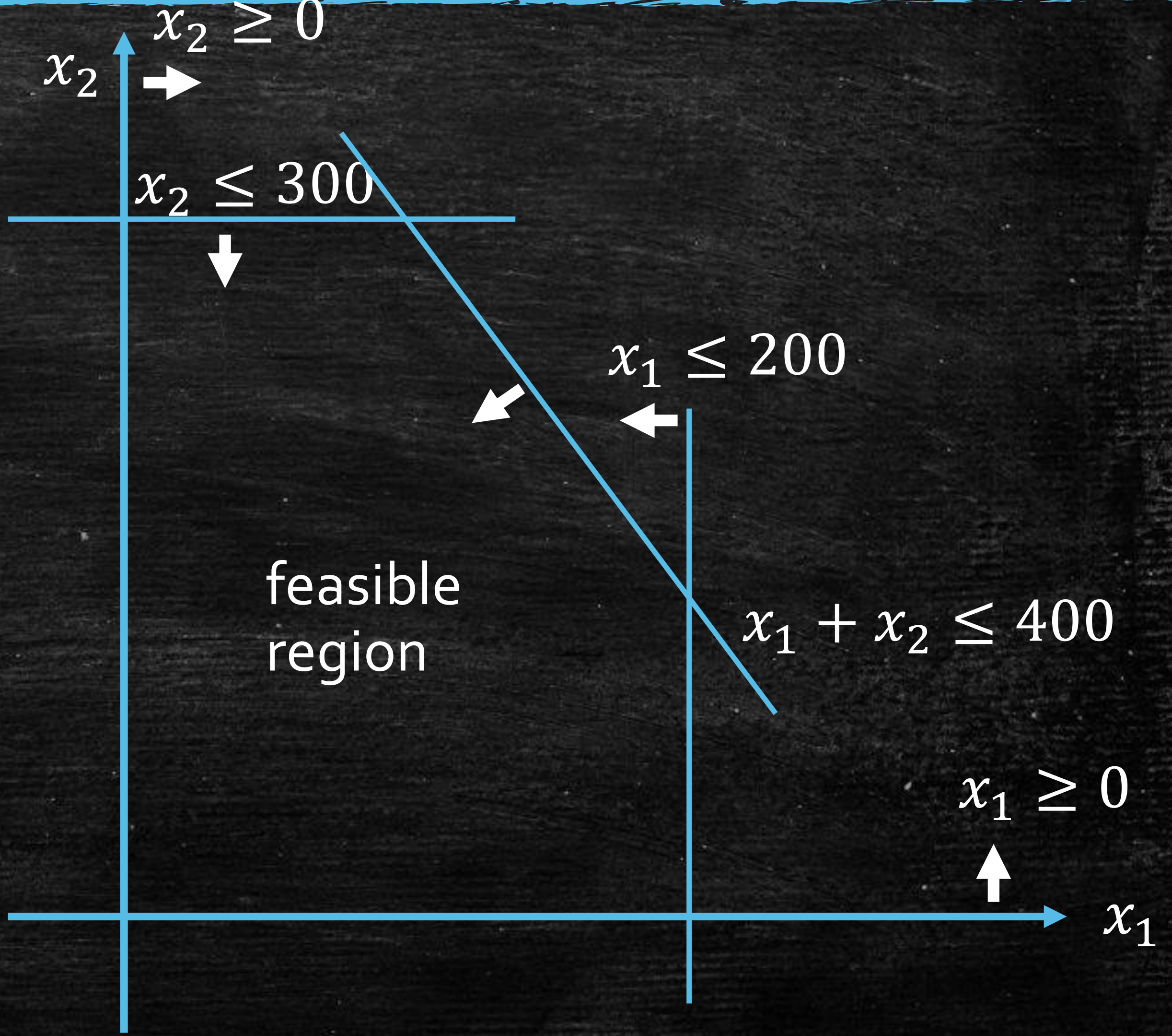
Example

- Suppose a factory can produce two kinds of products: oil and sugar.
- Profit for 1 tons of sugar: 1
- Profit for 1 tons of oil: 6
- Limited resources, can produce at most
  - 200 tons of sugar
  - 300 tons of oil
  - Overall weight is at most 400 tons
- Problem: maximize the profit

$$\begin{aligned} \text{maximize} \quad & x_1 + 6x_2 \\ \text{subject to} \quad & x_1 \leq 200 \\ & x_2 \leq 300 \\ & x_1 + x_2 \leq 400 \\ & x_1, x_2 \geq 0 \end{aligned}$$

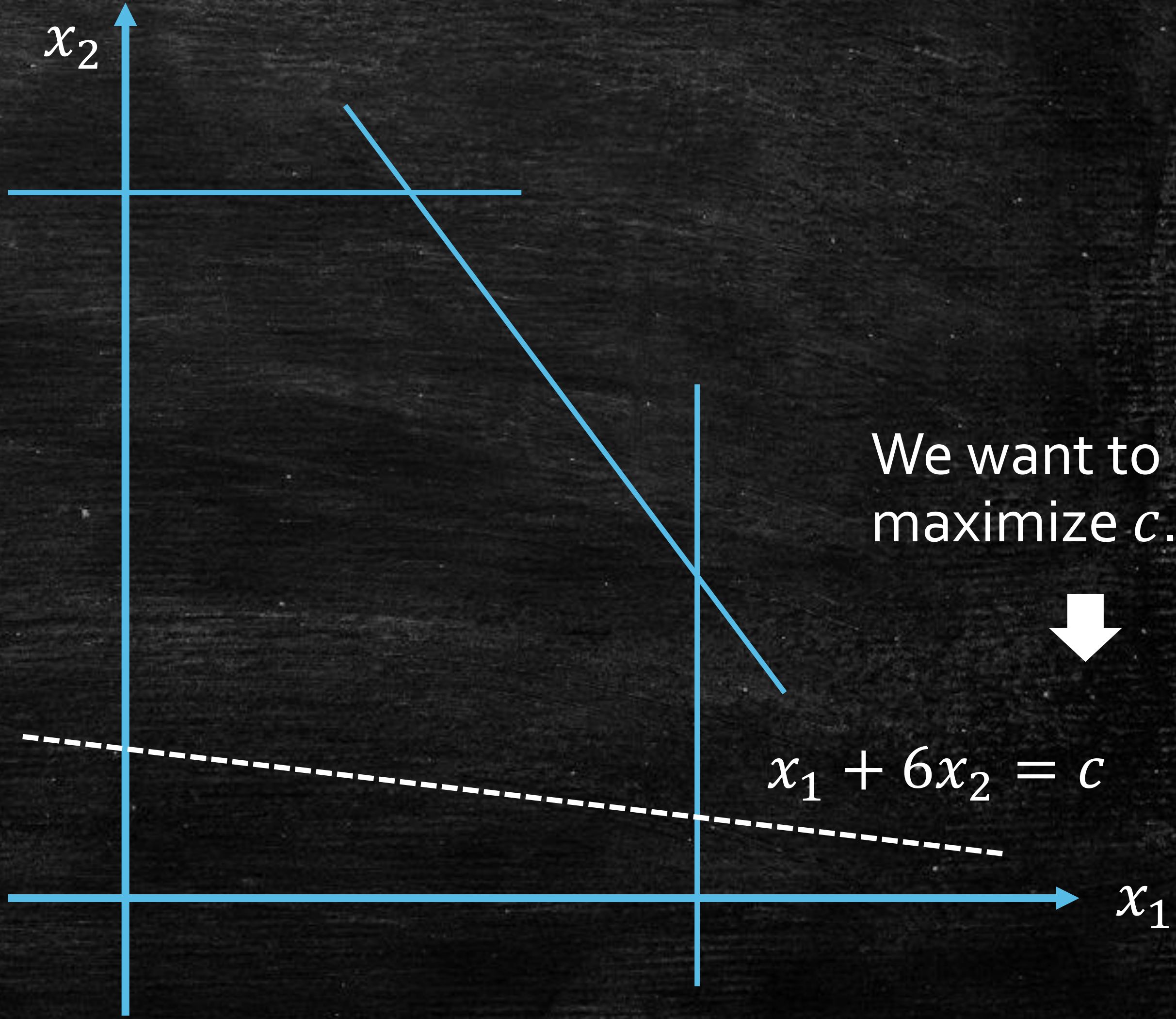
Feasible Region

$$\begin{aligned}
\text{maximize} \quad & x_1 + 6x_2 \\
\text{subject to} \quad & x_1 \leq 200 \\
& x_2 \leq 300 \\
& x_1 + x_2 \leq 400 \\
& x_1, x_2 \geq 0
\end{aligned}$$



Maximizing the Objective

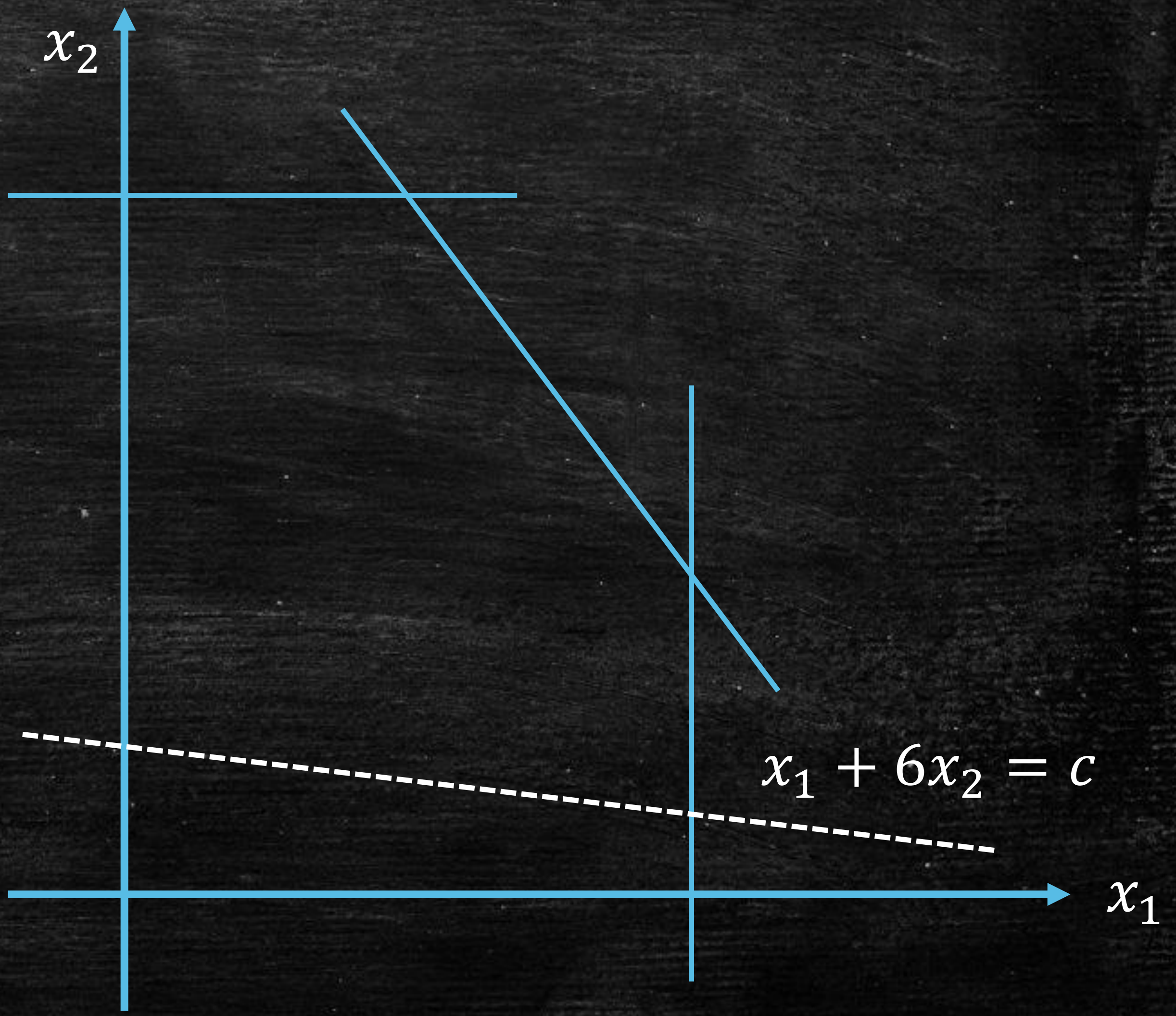
$$\begin{aligned} \text{maximize } \; & x_1 + 6x_2 \\ \text{subject to } \; & x_1 \le 200 \\ & x_2 \le 300 \\ & x_1 + x_2 \le 400 \\ & x_1, x_2 \ge 0 \end{aligned}$$



We want to maximize c.

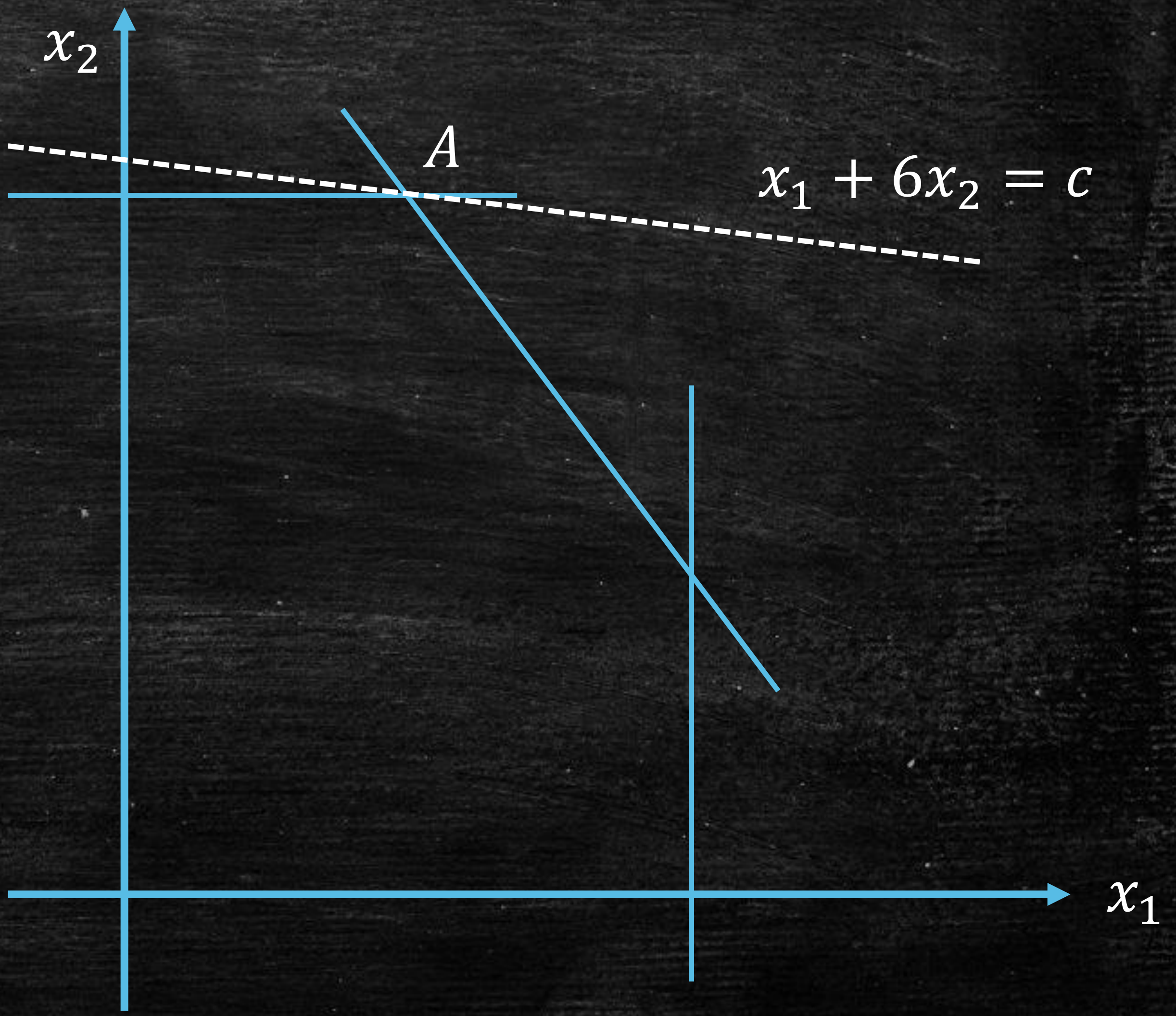
Maximizing the Objective

$$\begin{aligned} \text{maximize} \quad & x_1 + 6x_2 \\ \text{subject to} \quad & x_1 \leq 200 \\ & x_2 \leq 300 \\ & x_1 + x_2 \leq 400 \\ & x_1, x_2 \geq 0 \end{aligned}$$



Maximizing the Objective

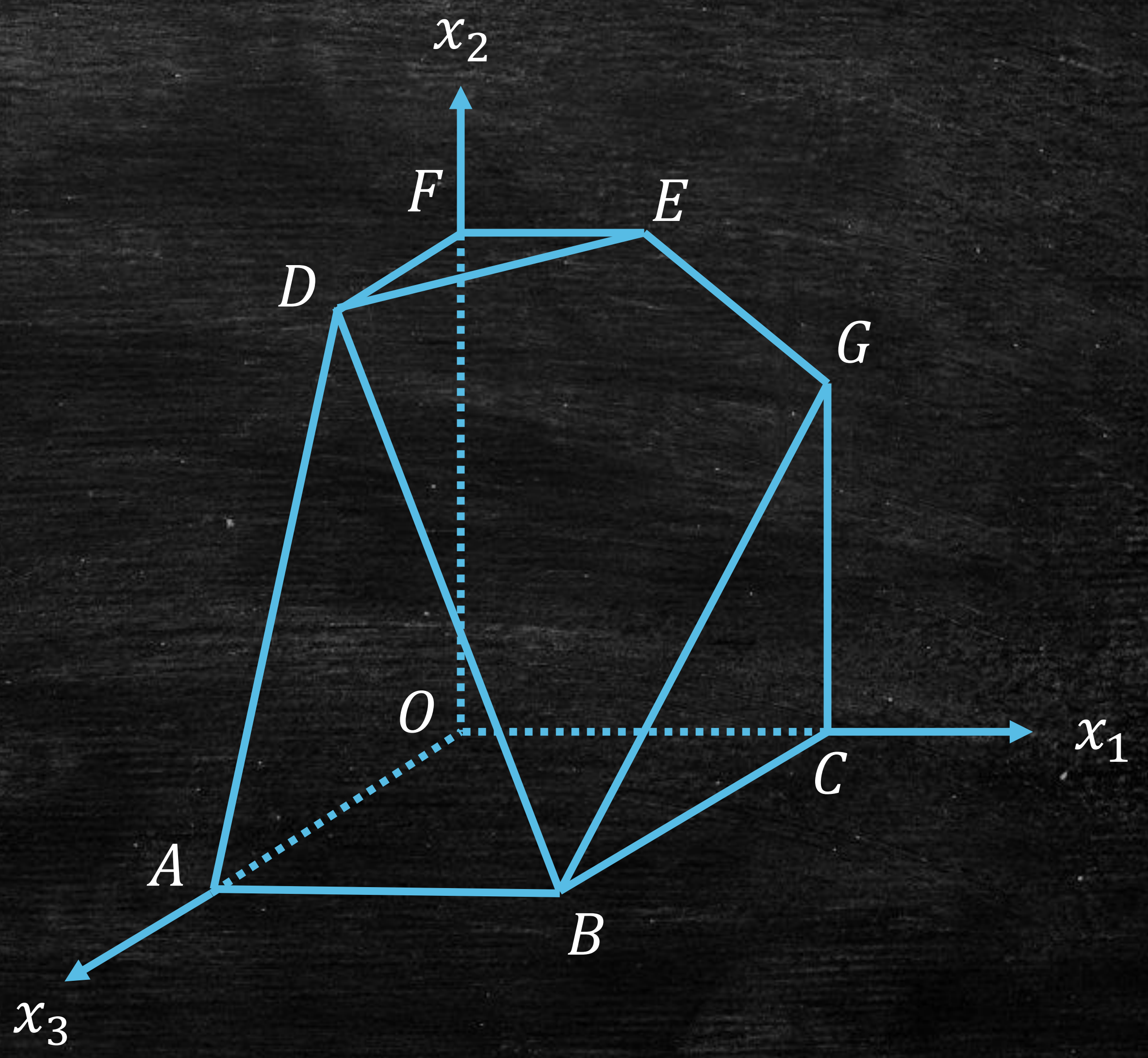
maximize $x_1 + 6x_2$
subject to $x_1 \leq 200$
$x_2 \leq 300$
$x_1 + x_2 \leq 400$
$x_1, x_2 \geq 0$



Optimum is obtained at vertex A, where $(x_1, x_2) = (100, 300)$ and $c = 1900$.

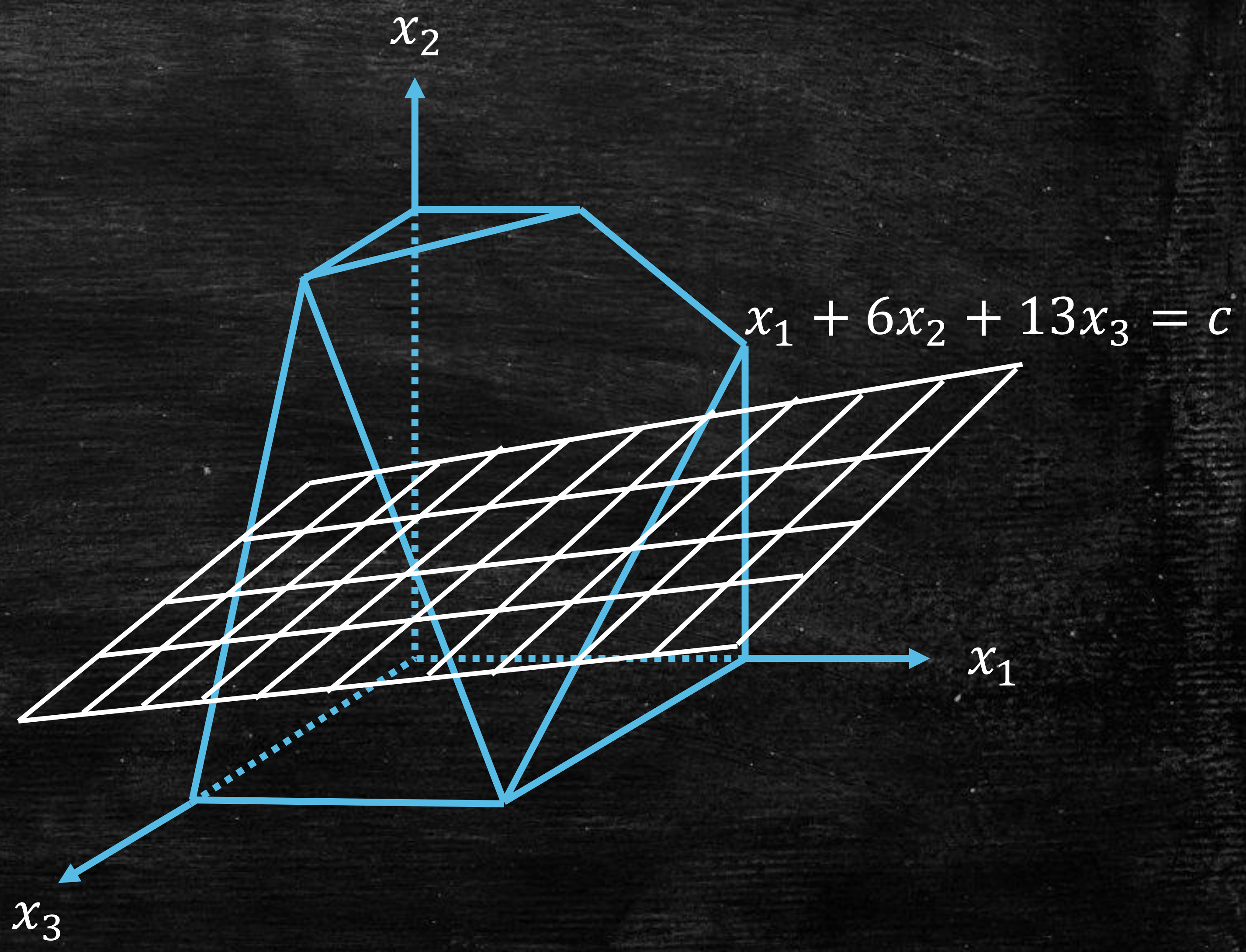
Another Example with Three variables

$$\begin{aligned}
\text{maximize} \quad & x_1 + 6x_2 + 13x_3 \\
\text{subject to} \quad & x_1 \le 200 \\
& x_2 \le 300 \\
& x_1 + x_2 + x_3 \le 400 \\
& x_2 + 3x_3 \le 600 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}$$



Another Example with Three variables

$$
\begin{aligned}
\text{maximize}\quad & x_1 + 6x_2 + 13x_3 \\
\text{subject to}\quad & x_1 \leq 200 \\
& x_2 \leq 300 \\
& x_1 + x_2 + x_3 \leq 400 \\
& x_2 + 3x_3 \leq 600 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$



Another Example with Three variables

maximize $x_1 + 6x_2 + 13x_3$

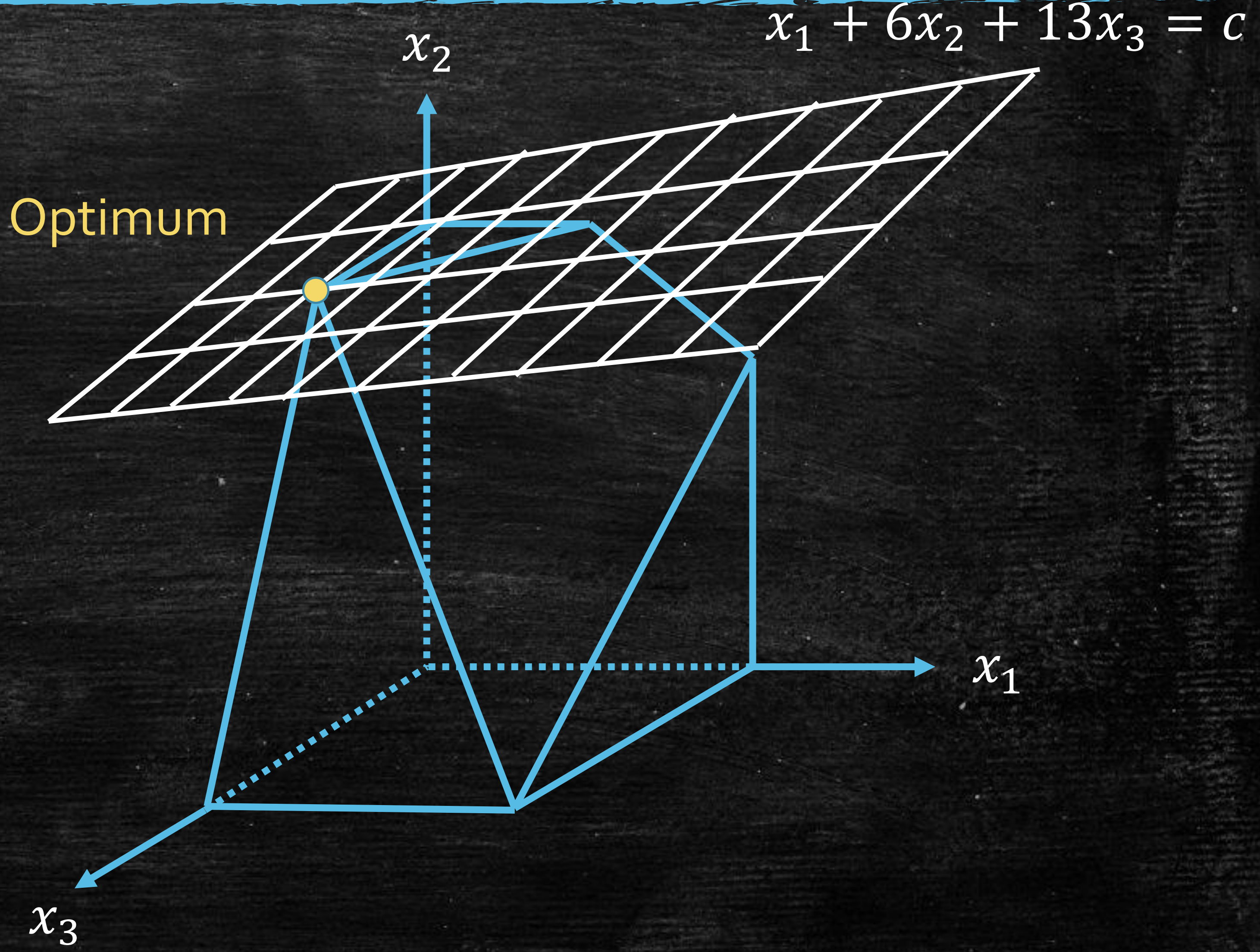
subject to $x_1 \leq 200$

$x_2 \leq 300$

$x_1 + x_2 + x_3 \leq 400$

$x_2 + 3x_3 \leq 600$

$x_1, x_2, x_3 \geq 0$



Important Observations

1. There always exists an optimum $\boldsymbol{x} = (x_1, \ldots, x_n)$ at a vertex of the polytope.
   - Linear objective $\Longrightarrow c = c_1 x_1 + \cdots + c_n x_n$ is a hyperplane.
   - Optimum is obtained only when the whole feasible region is below the hyperplane and the hyperplane “barely” intersect the region by a point.
2. The feasible region is always convex.
   - Linear Constraints $\Longrightarrow$ feasible region is bounded by hyperplanes.
3. A local maximum is also a global maximum.
   - By the convexity of the feasible region…

Simplex Method

- Choose an arbitrary starting vertex.
- Iteratively move to an adjacent vertex along an edge if such movement increase the objective.
- Terminate when we reach a local maximum.

Simplex Method

maximize $x_1 + 6x_2$

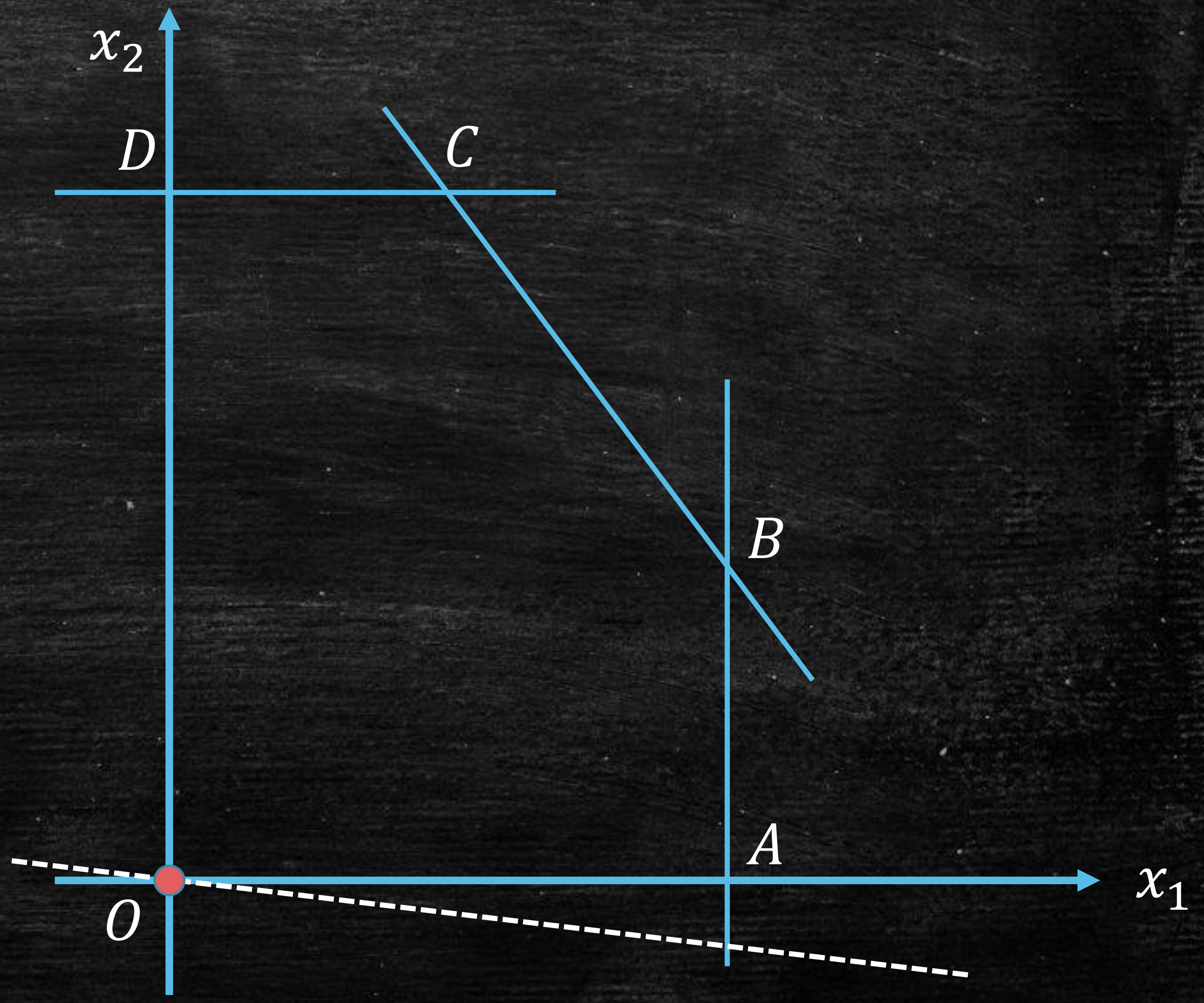
subject to $x_1 \leq 200$

$x_2 \leq 300$

$x_1 + x_2 \leq 400$

$x_1, x_2 \geq 0$

Starting from vertex O.



Simplex Method

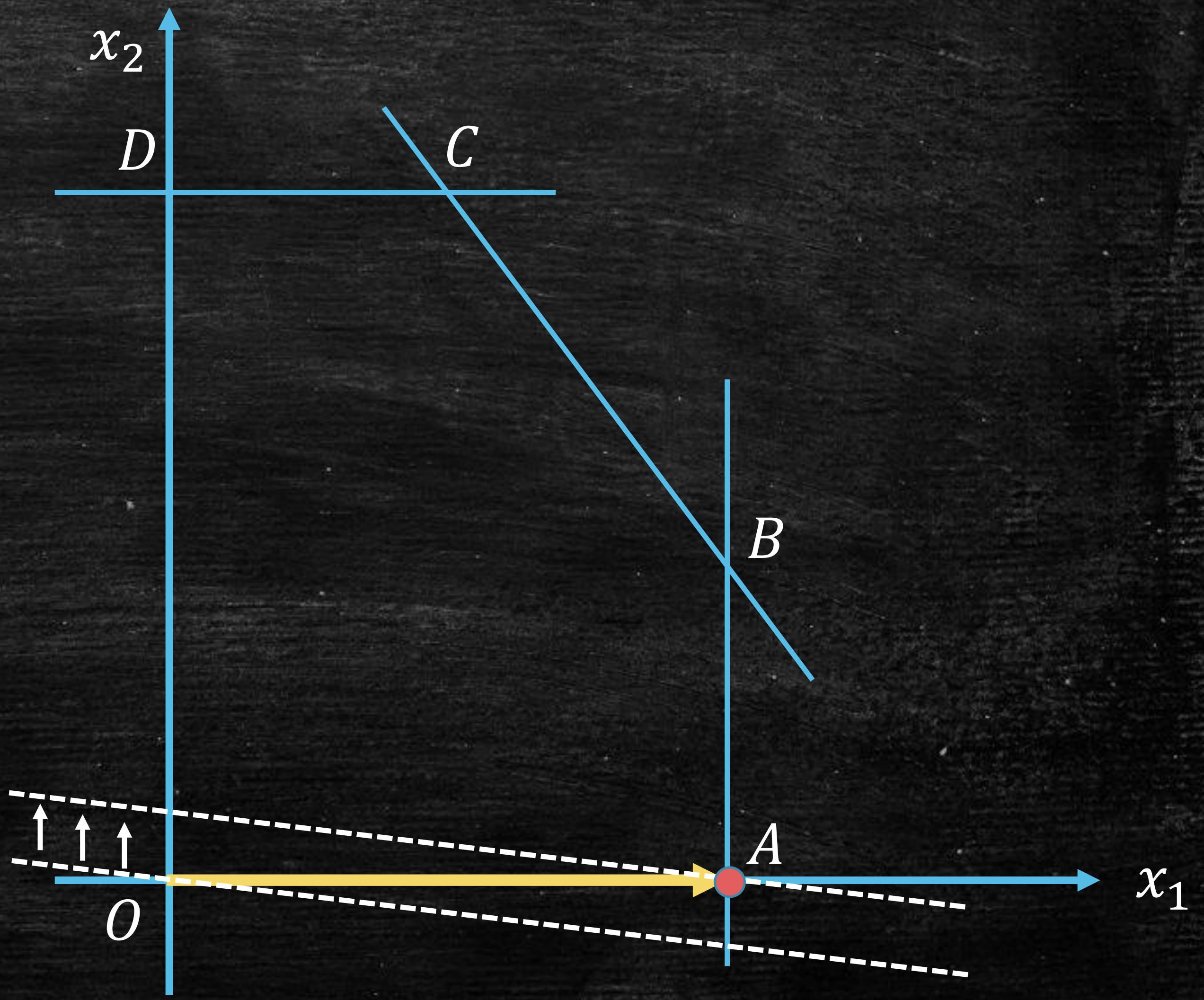
maximize $x_1 + 6x_2$

subject to $x_1 \leq 200$

$x_2 \leq 300$

$x_1 + x_2 \leq 400$

$x_1, x_2 \geq 0$

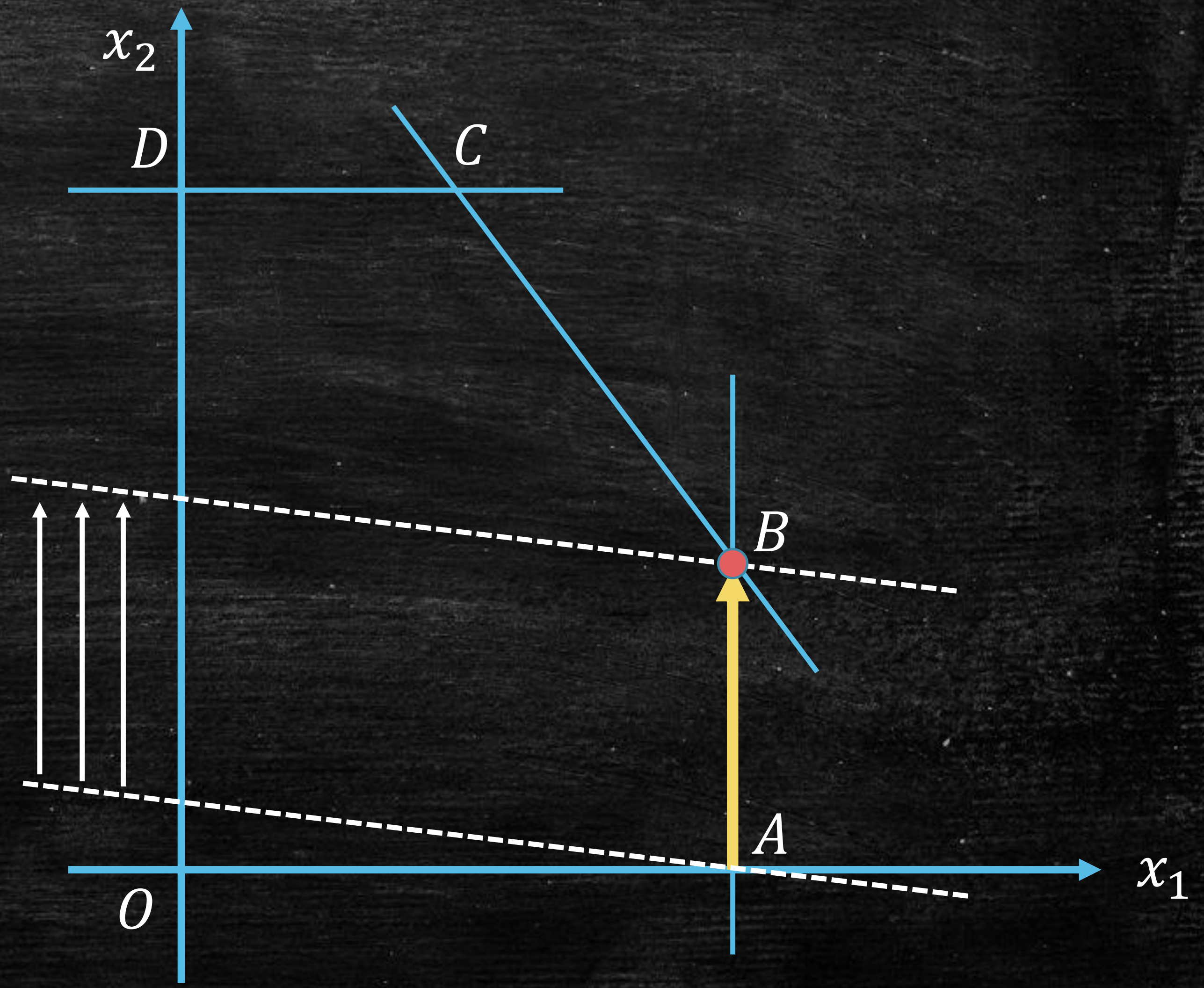


Moving from O to A increases the objective.

Simplex Method

$$\begin{aligned} \text{maximize} \quad & x_1 + 6x_2 \\ \text{subject to} \quad & x_1 \leq 200 \\ & x_2 \leq 300 \\ & x_1 + x_2 \leq 400 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Moving from A to B increases the objective.



Simplex Method

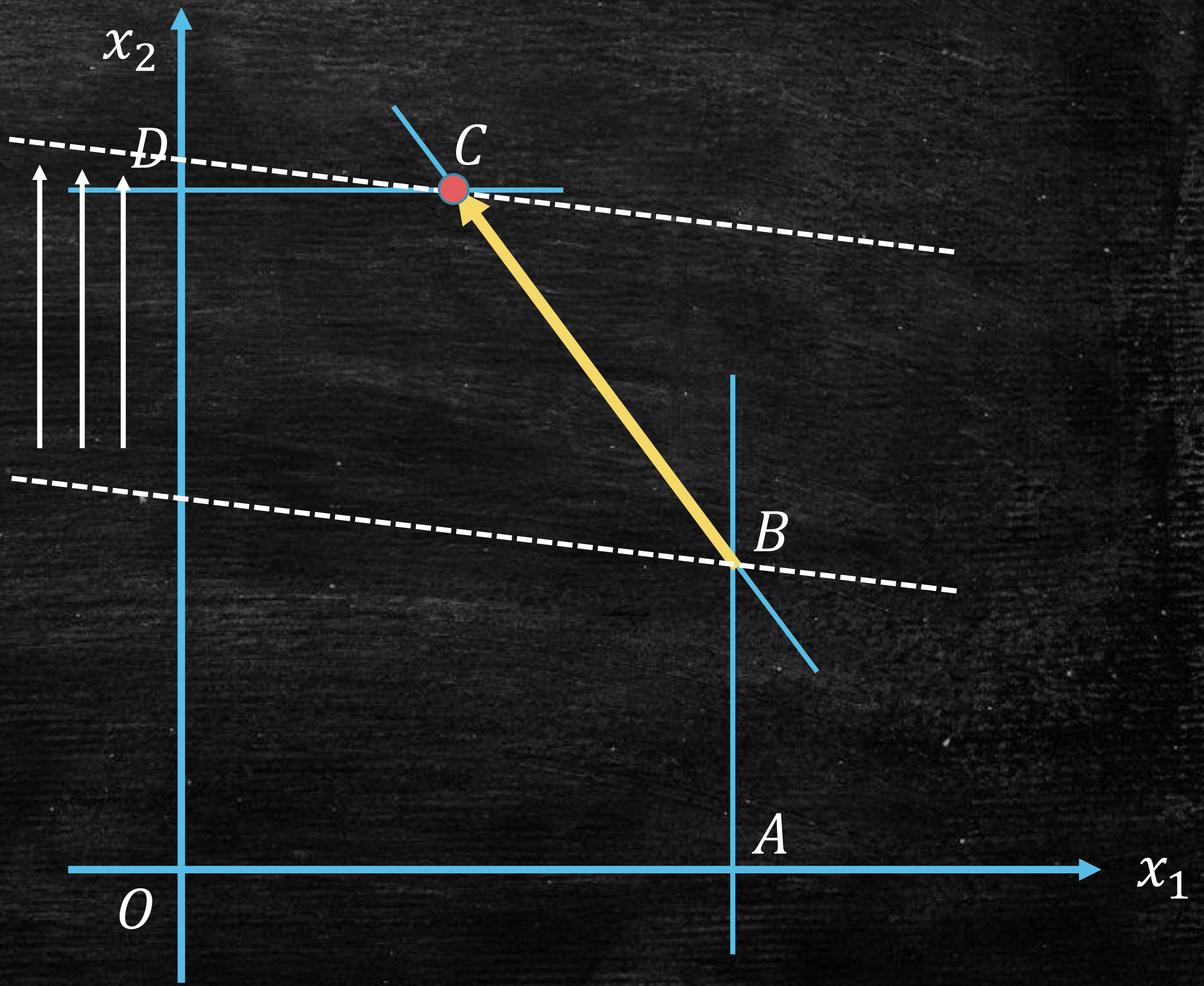
maximize $x_1 + 6x_2$

subject to $x_1 \leq 200$

$x_2 \leq 300$

$x_1 + x_2 \leq 400$

$x_1, x_2 \geq 0$

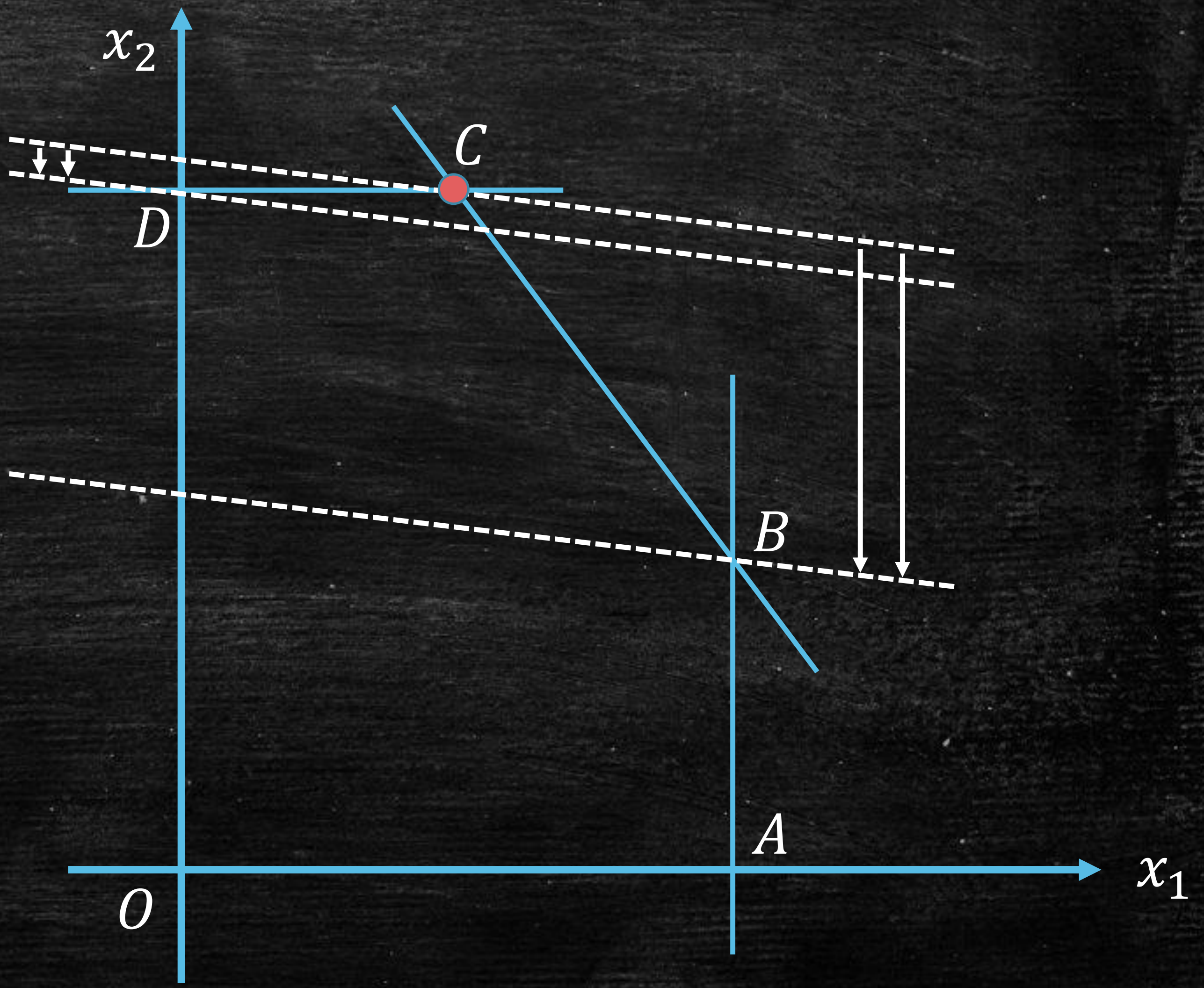


Moving from B to C increases the objective.

Simplex Method

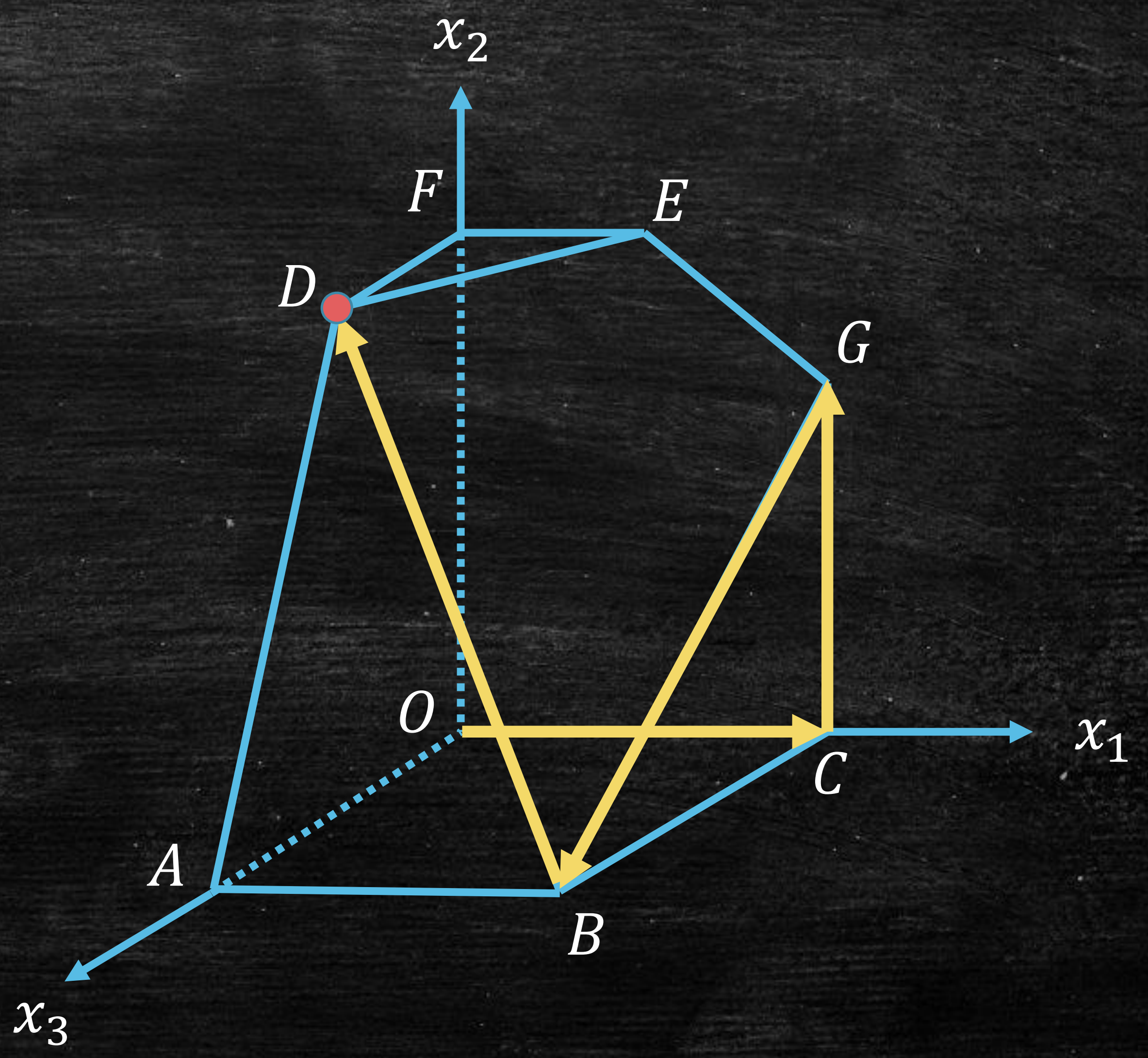
$$\begin{aligned} \text{maximize} \quad & x_1 + 6x_2 \\ \text{subject to} \quad & x_1 \le 200 \\ & x_2 \le 300 \\ & x_1 + x_2 \le 400 \\ & x_1, x_2 \ge 0 \end{aligned}$$

C is a local maximum:
Moving to either D or B decreases the objective.



Simplex Method

$$
\begin{aligned}
\text{maximize}\quad & x_1 + 6x_2 + 13x_3 \\
\text{subject to}\quad & x_1 \le 200 \\
& x_2 \le 300 \\
& x_1 + x_2 + x_3 \le 400 \\
& x_2 + 3x_3 \le 600 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}
$$



Some Details in Simplex Method

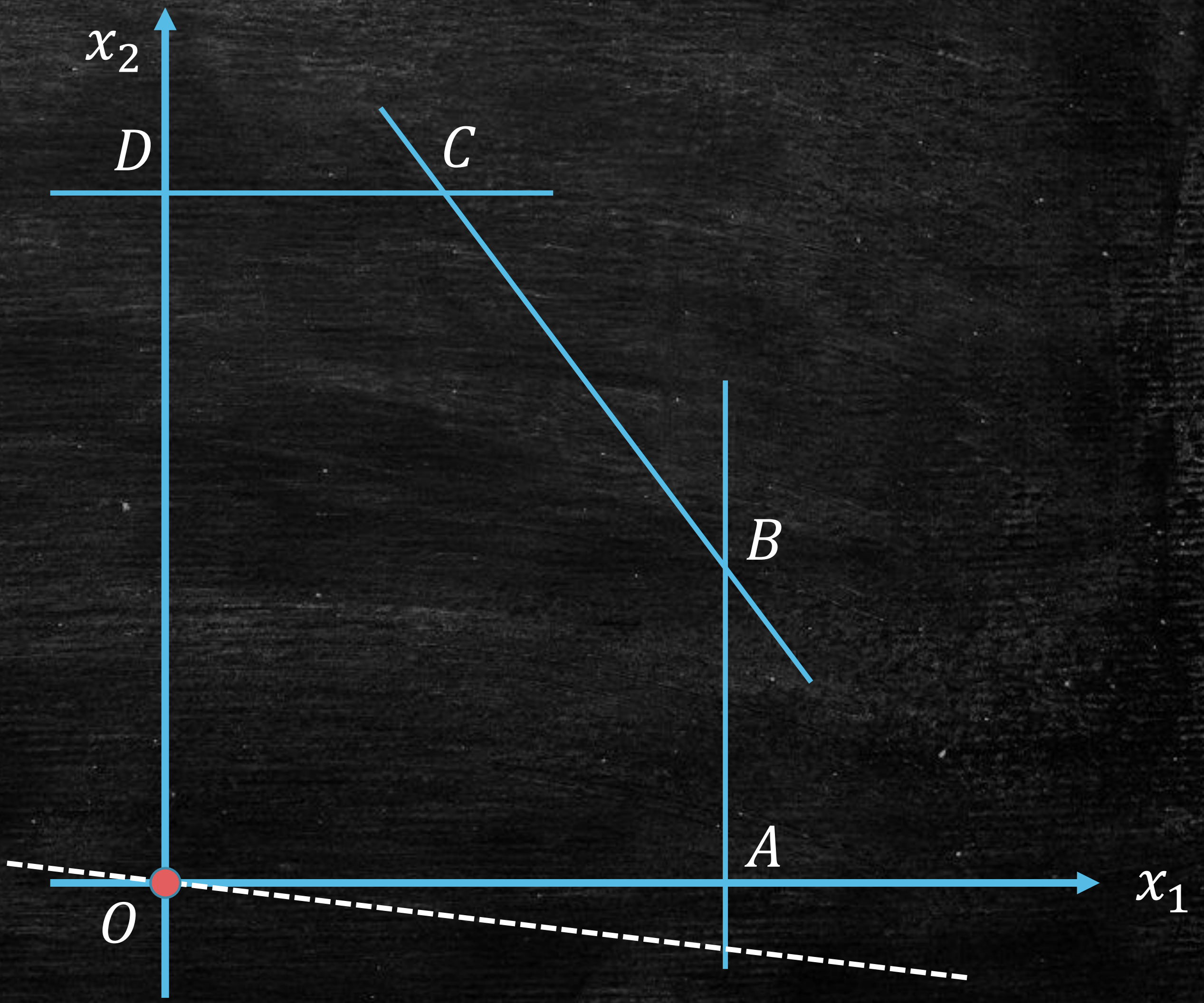
- What exactly is a vertex?
  - A point at the intersection of n linearly independent hyperplanes.
  - n hyperplanes intersect at exactly one point in $\mathbb{R}^n$
- What exactly is an edge?
  - The intersection of $n-1$ linearly independent hyperplanes.
  - $n-1$ hyperplanes intersect at a line in $\mathbb{R}^n$
- How do we "move from one vertex to another adjacent vertex along an edge"?
  - Relax one of the n constraint and impose another.
  - The new vertex can be computed by solving a system of n linear equations.

Simplex Method

$$\begin{aligned} \text{maximize} \quad & x_1 + 6x_2 \\ \text{subject to} \quad & x_1 \leq 200 \\ & x_2 \leq 300 \\ & x_1 + x_2 \leq 400 \\ & x_1, x_2 \geq 0 \end{aligned}$$

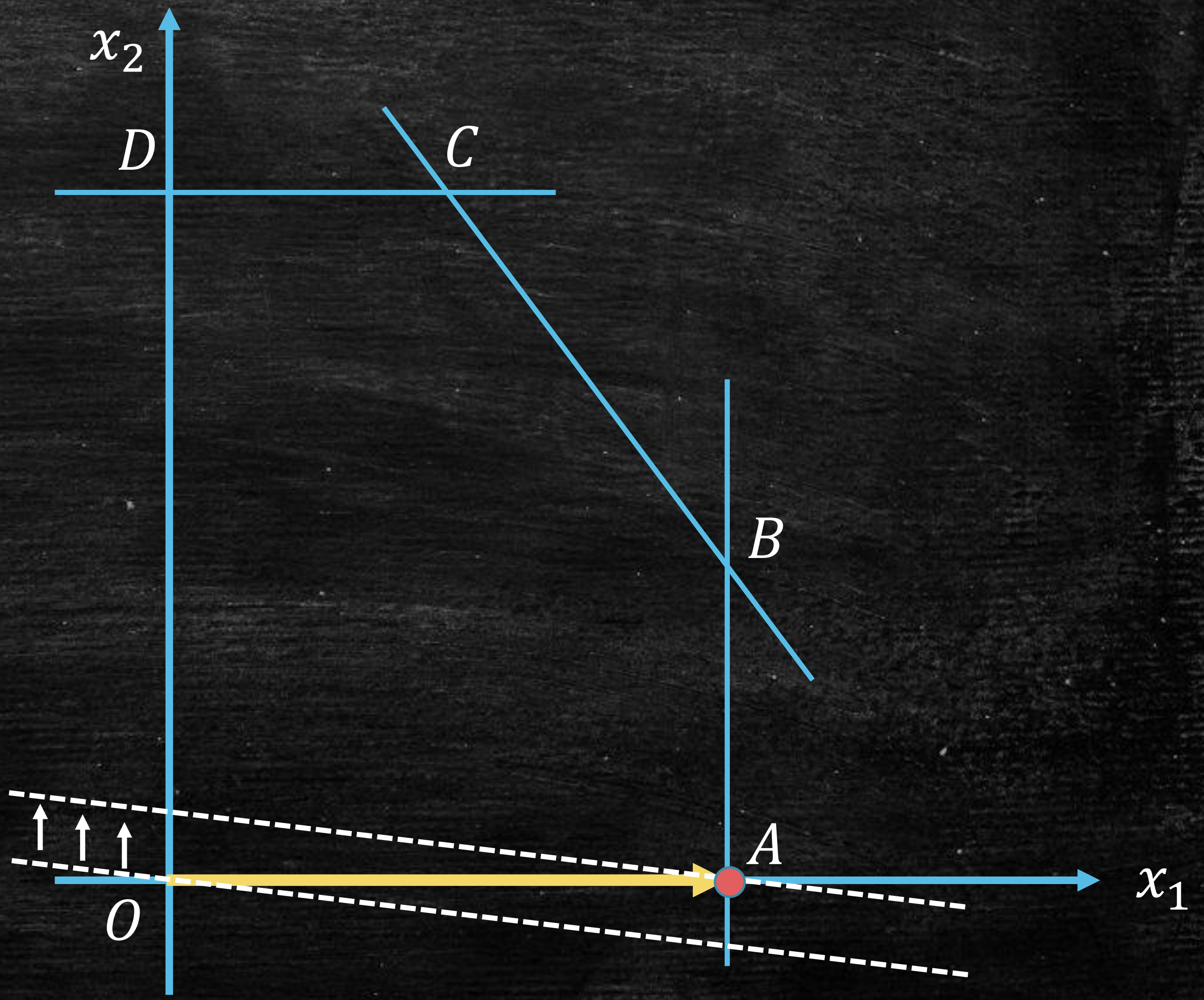
Starting from vertex O:

- Intersection of two lines $x_1 = 0$ and $x_2 = 0$.



Simplex Method

$$\begin{aligned} \text{maximize} \quad & x_1 + 6x_2 \\ \text{subject to} \quad & x_1 \le 200 \\ & x_2 \le 300 \\ & x_1 + x_2 \le 400 \\ & x_1, x_2 \ge 0 \end{aligned}$$

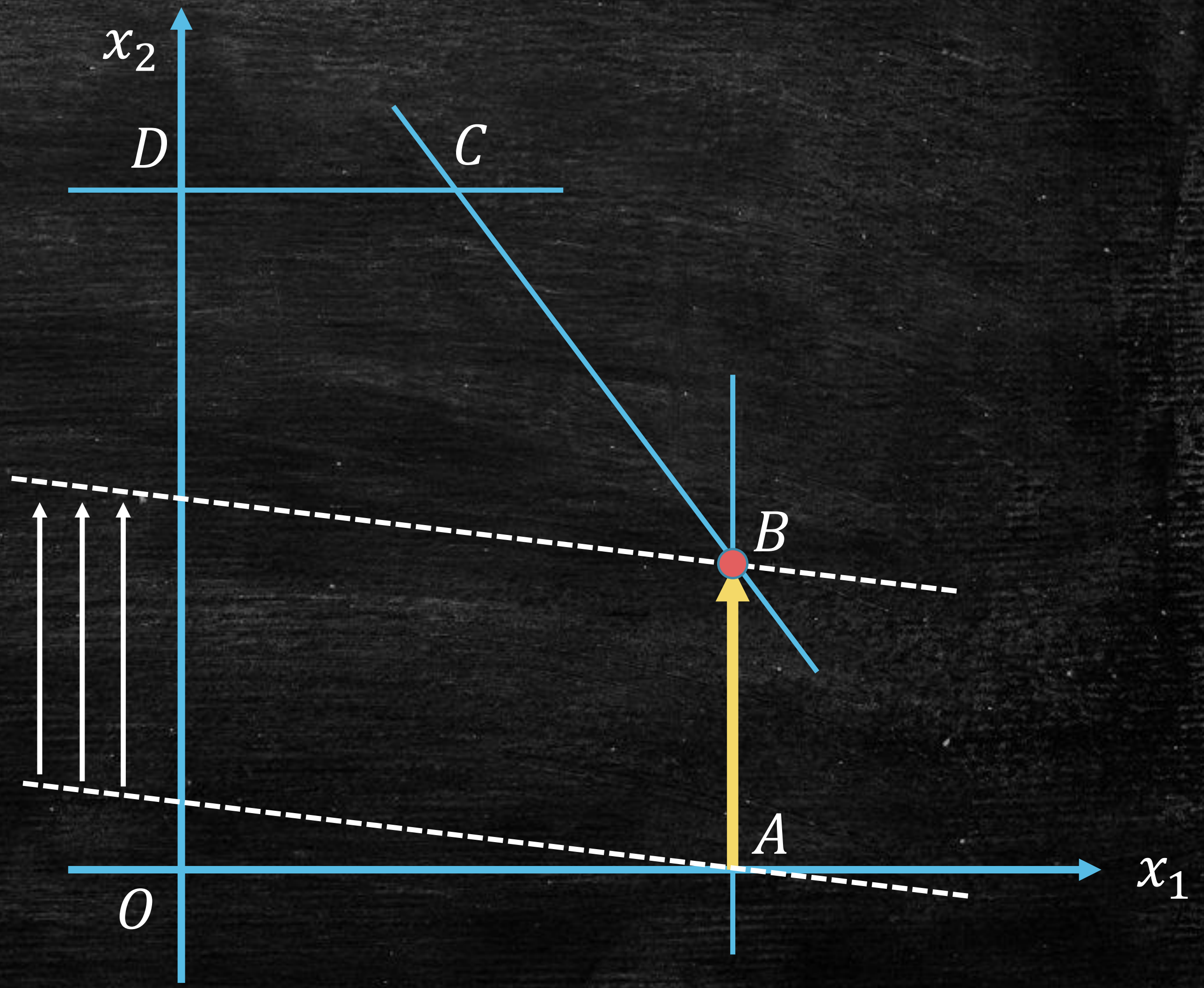


Moving from O to A:

- Relax $x_1 = 0$ and impose $x_1 = 200$
- $\begin{cases} x_1 = 200 \\ x_2 = 0 \end{cases}$

Simplex Method

$$\begin{aligned} \text{maximize} \ & x_1 + 6x_2 \\ \text{subject to} \ & x_1 \le 200 \\ & x_2 \le 300 \\ & x_1 + x_2 \le 400 \\ & x_1, x_2 \ge 0 \end{aligned}$$

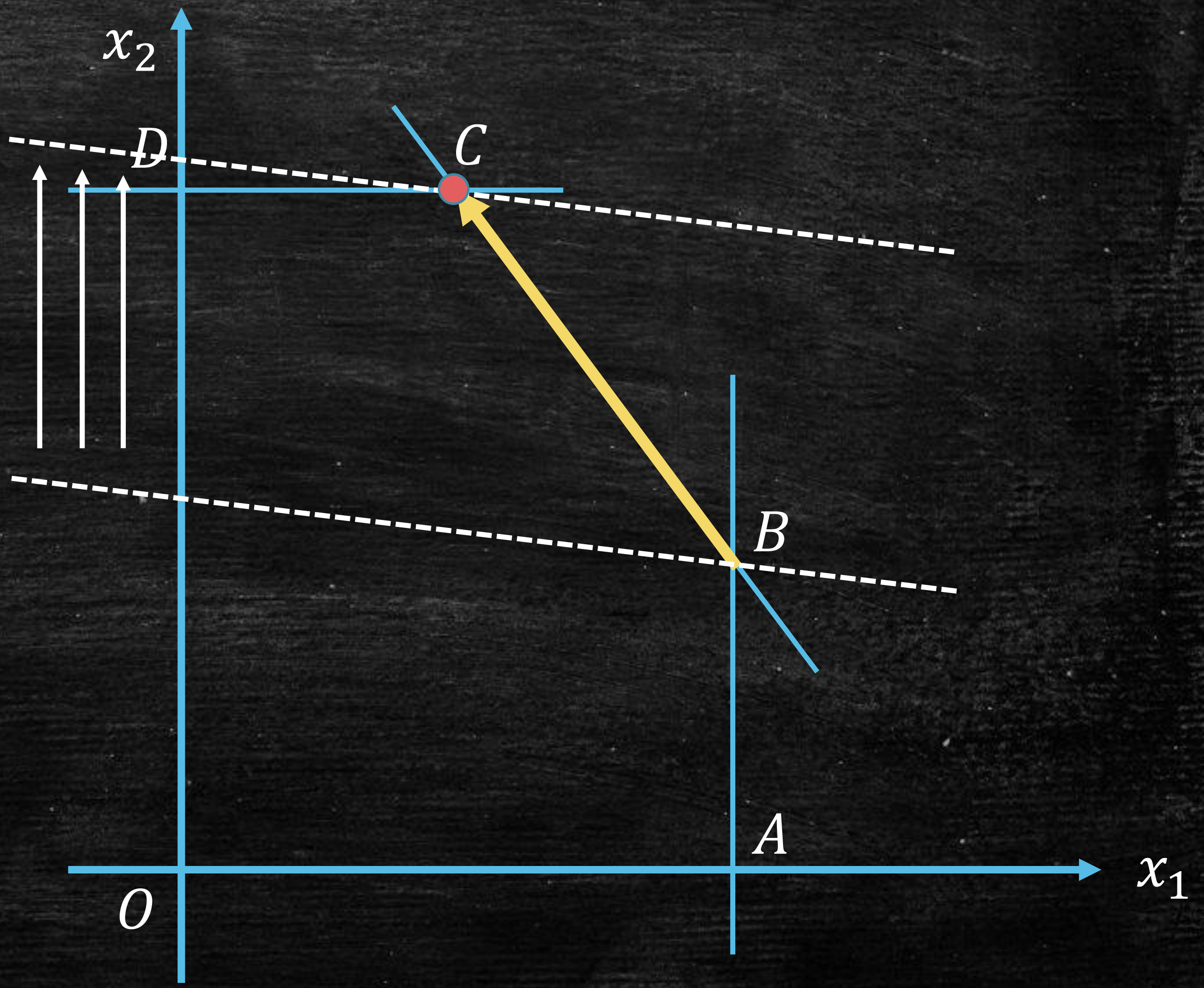


Moving from A to B:

- Relax $x_2 = 0$ and impose $x_1 + x_2 = 400$
- $\begin{cases} x_1 + x_2 = 400 \\ x_1 = 200 \end{cases}$

Simplex Method

$$\begin{aligned} \text{maximize} \quad & x_1 + 6x_2 \\ \text{subject to} \quad & x_1 \leq 200 \\ & x_2 \leq 300 \\ & x_1 + x_2 \leq 400 \\ & x_1, x_2 \geq 0 \end{aligned}$$

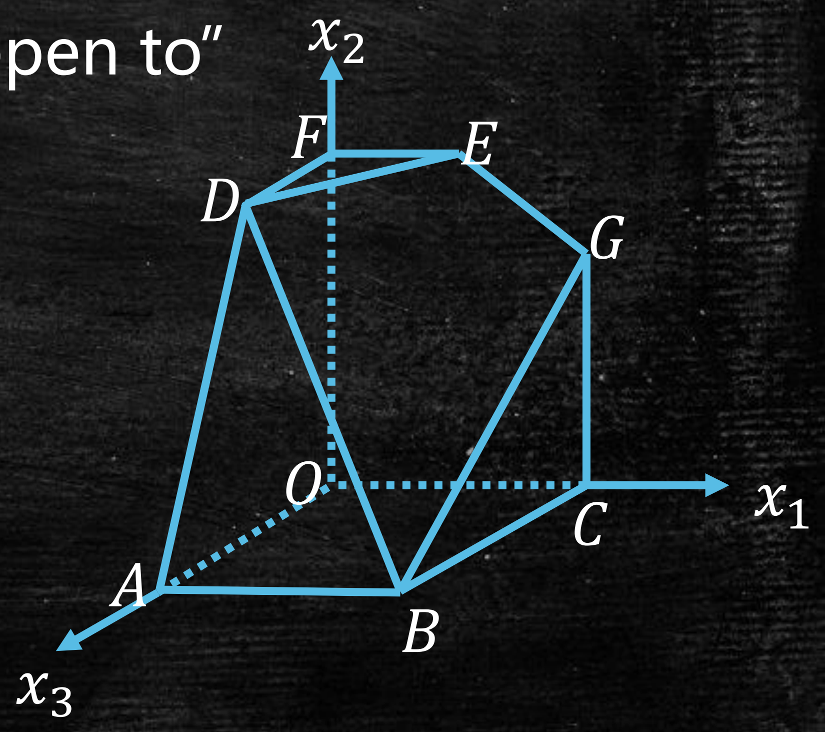


Moving from B to C:

- Relax $x_1 = 200$ and impose $x_2 = 300$
- $\begin{cases} x_1 + x_2 = 400 \\ \quad x_2 = 300 \end{cases}$

Missing Details not Covered in This Lecture…

- How to find a starting vertex?
- How to find a neighbor that guarantees increment to objective?
- Degenerated vertex: $n+1$ hyperplanes "happen to" intersect at a single point.
  - E.g., Vertex B and D
- Unbounded feasible region…
- And many more…

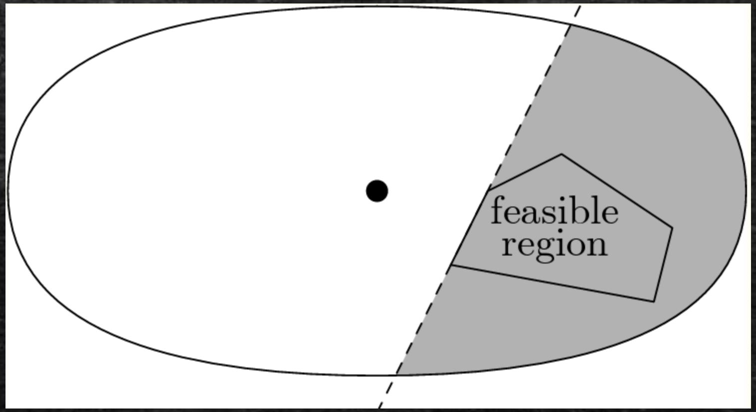


Time Complexity for Simplex Method

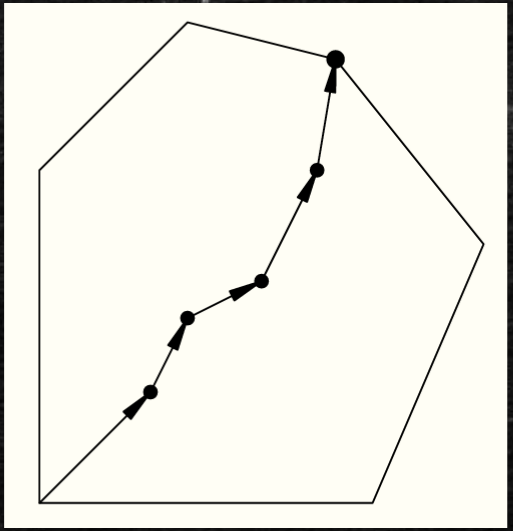
- There are exponentially many vertices: $\binom{m}{n}$ for m constraints and n variables.
- Worst-case running time: exponential
  - Many attempts have failed.
  - e.g., choose neighbors with highest objective value, choose neighbors randomly, etc.
- [Teng & Spielman] Smoothed analysis
  - Average case polynomial time if add random Gaussian noise to the constraints.
- Runs fast in practice, and most commonly used

Polynomial Time Algorithms for LP

- Ellipsoid Method



- Interior Point Method



Standard Form LP

- Maximization as objective with "≤" constraints and non-negative variables.

$$
\begin{array}{ll}
\text{maximize} & c_1 x_1 + c_2 x_2 + \cdots c_n x_n \\
\text{subject to} & a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \le b_1 \\
& a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \le b_2 \\
& \vdots \\
& a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \le b_m \\
& x_1, x_2, \ldots, x_n \ge 0
\end{array}
$$

$$
\begin{array}{ll}
\text{maximize} & \mathbf{c}^\top \mathbf{x} \\
\text{subject to} & A\mathbf{x} \le \mathbf{b} \\
& \mathbf{x} \ge \mathbf{0}
\end{array}
$$

Other Forms Reduce to Standard Form

- Minimization to Maximization
  - $\min c_1x_1 + \cdots + c_nx_n \quad \Leftrightarrow \quad \max -c_1x_1 - \cdots - c_nx_n$
- $\geq$-inequalities
  - $a_1x_1 + \cdots + a_nx_n \geq b \quad \Leftrightarrow \quad -a_1x_1 - \cdots - a_nx_n \leq -b$
- Inequality $\Leftrightarrow$ Equality
  - $a_1x_1 + \cdots + a_nx_n = b \quad \Leftrightarrow \quad \begin{cases} a_1x_1 + \cdots + a_nx_n \leq b \\ a_1x_1 + \cdots + a_nx_n \geq b \end{cases}$
  - $a_1x_1 + \cdots + a_nx_n \leq b \quad \Leftrightarrow \quad a_1x_1 + \cdots + a_nx_n + s = b$
- Variable with unrestricted signs
  - Introduce two variables x^+ and x^- with standard constraints $x^+, x^- \geq 0$
  - Replace x with $x^+ - x^-$

Take-Home Message

- A linear program can be solved in a polynomial time.
- Whenever a problem can be formulated by a linear program, it is polynomial-time solvable.

Formulation as Linear Program

- The maximum flow problem can be formulated by a linear program.

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{u:(s,u)\in E} f_{su} & \\ \text{subject to} & 0 \le f_{uv} \le c_{uv} & \forall (u,v) \in E \\ & \displaystyle\sum_{v:(v,u)\in E} f_{vu} = \sum_{w:(v,w)\in E} f_{uw} & \forall u \in V \setminus \{s,t\} \end{array}$$

- Ford-Fulkerson Method implements the simplex method.

Formulation as Linear Program

- The "highway driving" problem in Assignment 3 can be formulated as a linear program.
- Capacity of tank: C
- Location and unit price of i-th station: d_i, p_i
- Start: 0-th station Destination: n-th station

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_i p_i x_i & \\
\text{subject to} & y_0 = 0 & \\
 & y_i = y_{i-1} + x_{i-1} - (d_i - d_{i-1}) & \text{for } i = 1, \dots, n \\
 & x_i + y_i \leq C & \text{for } i = 0, 1, \dots, n \\
 & x_0, x_1 \dots, x_n, y_0, y_1, \dots, y_n \geq 0 &
\end{array}
$$

Part II:
LP Duality

Motivation

- We have seen that the optimal solution for the LP below is $(x_1,\ x_2) = (100, 300)$, with value 1900.
  - Geometric argument, argument based on simplex method
- Let's try to prove it by some simple observations from the LP itself!

$$\begin{aligned} \text{maximize} \quad & x_1 + 6x_2 \\ \text{subject to} \quad & x_1 \le 200 \\ & x_2 \le 300 \\ & x_1 + x_2 \le 400 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Motivation

$$
\begin{aligned}
\text{maximize} \quad & x_1 + 6x_2 \\
\text{subject to} \quad & x_1 \le 200 && \text{(i)} \\
& x_2 \le 300 && \text{(ii)} \\
& x_1 + x_2 \le 400 && \text{(iii)} \\
& x_1, x_2 \ge 0
\end{aligned}
$$

- Let's try adding (i) to 6 times (ii): $x_1 + 6x_2 \le 200 + 6 \times 300 = 2000$
- We know that any solution (x_1, x_2) cannot yield objective value greater than 2000.
- Can we combine the inequality in a better way to show that the objective value is at most 1900?

Motivation

$$\begin{aligned} \text{maximize } & x_1 + 6x_2 & \\ \text{subject to } & x_1 \leq 200 & \text{(i)} \\ & x_2 \leq 300 & \text{(ii)} \\ & x_1 + x_2 \leq 400 & \text{(iii)} \\ & x_1, x_2 \geq 0 & \end{aligned}$$

- Can we combine the inequality in a better way to show that the objective value is at most 1900?
- Yes, we can:
  - Multiple (ii) by 5 and add to (iii): $x_1 + 6x_2 \leq 300 \times 5 + 400 = 1900.$
- This proves that $(x_1, x_2) = (100, 300)$ with objective value 1900 is optimal!

Let's try this one...

- Suppose we multiple (i) by y_1, (ii) by y_2, (iii) by y_3, and (iv) by y_4.
- We have $(y_1 + y_3)x_1 + (y_2 + y_3 + y_4)x_2 + (y_3 + 3y_4)x_3 \leq 200y_1 + 300y_2 + 400y_3 + 600y_4$.
- We need $y_1, y_2, y_3, y_4 \geq 0$ to keep the inequality.
- To find an upper bound to the objective $x_1 + 6x_2 + 13x_3$, we need to make sure $x_1 + 6x_2 + 13x_3 \leq (y_1 + y_3)x_1 + (y_2 + y_3 + y_4)x_2 + (y_3 + 3y_4)x_3$ holds for every (x_1, x_2, x_3).
- Since $x_1, x_2, x_3 \geq 0$, we must have:
  - $y_1 + y_3 \geq 1$
  - $y_2 + y_3 + y_4 \geq 6$
  - $y_3 + 3y_4 \geq 13$

$$
\begin{aligned}
\text{maximize} \quad & x_1 + 6x_2 + 13x_3 \\
\text{subject to} \quad & x_1 \leq 200 && \text{(i)} \\
& x_2 \leq 300 && \text{(ii)} \\
& x_1 + x_2 + x_3 \leq 400 && \text{(iii)} \\
& x_2 + 3x_3 \leq 600 && \text{(iv)} \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Let's try this one…

- $(y_1 + y_3)x_1 + (y_2 + y_3 + y_4)x_2 + (y_3 + 3y_4)x_3 \le 200y_1 + 300y_2 + 400y_3 + 600y_4$.
- Since $x_1, x_2, x_3 \ge 0$, we must have:
  - $y_1 + y_3 \ge 1$
  - $y_2 + y_3 + y_4 \ge 6$
  - $y_3 + 3y_4 \ge 13$
- Now, we want to find the tightest possible upper-bound to $x_1 + 6x_2 + 13x_3$.
- This means we want to minimize $200y_1 + 300y_2 + 400y_3 + 600y_4$.

$$
\begin{array}{lll}
\text{maximize} & x_1 + 6x_2 + 13x_3 & \\
\text{subject to} & x_1 \le 200 & \text{(i)} \\
& x_2 \le 300 & \text{(ii)} \\
& x_1 + x_2 + x_3 \le 400 & \text{(iii)} \\
& x_2 + 3x_3 \le 600 & \text{(iv)} \\
& x_1, x_2, x_3 \ge 0 &
\end{array}
$$

Dual Program

- The problem of finding the tightest upper-bound can be formulated by another linear program!
- This linear program is called the dual program, and the original one is called the primal program.

$$
\begin{aligned}
\text{maximize} \quad & x_1 + 6x_2 + 13x_3 \\
\text{subject to} \quad & x_1 \leq 200 \\
& x_2 \leq 300 \\
& x_1 + x_2 + x_3 \leq 400 \\
& x_2 + 3x_3 \leq 600 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{minimize} \quad & 200y_1 + 300y_2 + 400y_3 + 600y_4 \\
\text{subject to} \quad & y_1 + y_3 \geq 1 \\
& y_2 + y_3 + y_4 \geq 6 \\
& y_3 + 3y_4 \geq 13 \\
& y_1, y_2, y_3, y_4 \geq 0
\end{aligned}
$$

Dual Program

- Factory Example:

$$\begin{aligned} \text{maximize} \quad & x_1 + 6x_2 \\ \text{subject to} \quad & x_1 \leq 200 \\ & x_2 \leq 300 \\ & x_1 + x_2 \leq 400 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & 200y_1 + 300y_2 + 400y_3 \\ \text{subject to} \quad & y_1 + y_3 \geq 1 \\ & y_2 + y_3 \geq 6 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

- Dual program for standard form:

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & \mathbf{b}^\top \mathbf{y} \\ \text{subject to} \quad & \mathbf{y}^\top A \geq \mathbf{c}^\top \\ & \mathbf{y} \geq \mathbf{0} \end{aligned}$$

Weak Duality Theorem

- By our motivation of dual program, we obtain the following theorem.
- Theorem [Weak Duality Theorem]. If $\hat{\mathbf{x}}$ is a feasible solution to (a) and $\hat{\mathbf{y}}$ is a feasible solution to (b), then $\mathbf{c}^\top \hat{\mathbf{x}} \leq \mathbf{b}^\top \hat{\mathbf{y}}$.

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \qquad \text{(a)}$$

$$\begin{array}{ll} \text{minimize} & \mathbf{b}^\top \mathbf{y} \\ \text{subject to} & \mathbf{y}^\top A \geq \mathbf{c}^\top \\ & \mathbf{y} \geq \mathbf{0} \end{array} \qquad \text{(b)}$$

Strong Duality Theorem: This gap is always closed!

Strong Duality Theorem

- Theorem [Strong Duality Theorem]. Let $\mathbf{x}^*$ be the optimal solution to (a) and $\mathbf{y}^*$ be the optimal solution to (b), then $\mathbf{c}^\top \mathbf{x}^* = \mathbf{b}^\top \mathbf{y}^*$.

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \le \mathbf{b} \qquad \text{(a)} \\ & \mathbf{x} \ge \mathbf{0} \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & \mathbf{b}^\top \mathbf{y} \\ \text{subject to} \quad & \mathbf{y}^\top A \ge \mathbf{c}^\top \qquad \text{(b)} \\ & \mathbf{y} \ge \mathbf{0} \end{aligned}$$

Primal feasible | Primal OPT = Dual OPT | Dual feasible

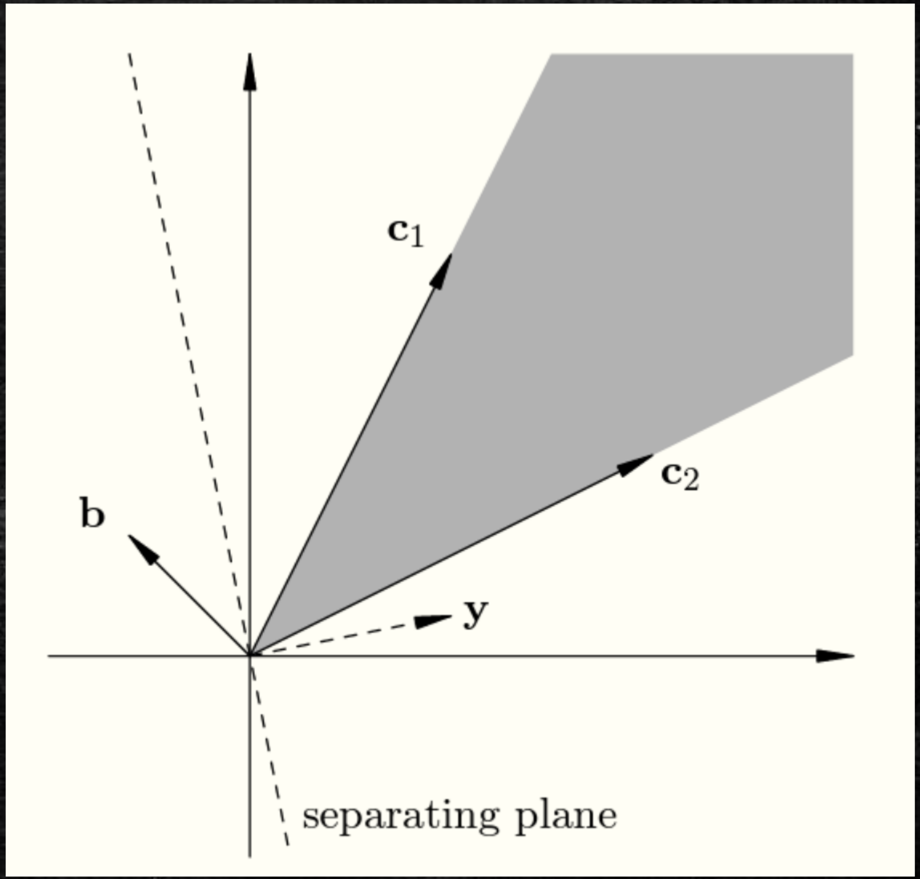
Application of Strong Duality Theorem

- Max-Flow-Min-Cut Theorem
- Minimax Theorem
- Kőnig-Egerváry Theorem
- Design approximation algorithms:
  - Dual fitting
  - Primal-Dual Schema
- Economic interpretation: “resource allocation”-”resource valuation”

Proof of Strong Duality Theorem

- **Theorem [Farkas Lemma].** Exactly one of the followings holds for matrix $A \in \mathbb{R}^{m \times n}$ and vector $\mathbf{b} \in \mathbb{R}^m$:
  1. There exists $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \geq \mathbf{0}$ such that $A\mathbf{x} = \mathbf{b}$.
  2. There exists $\mathbf{y} \in \mathbb{R}^m$ such that $A^\top \mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}^\top \mathbf{y} < 0$.
- $\{A\mathbf{x} \mid \mathbf{x} \geq \mathbf{0}\}$ is the grey area.
- 1 says that $\mathbf{b}$ is inside the grey area.
- 2 says that we can separate the grey area and $\mathbf{b}$ by a hyperplane (defined by the normal vector $\mathbf{y}$).
  - In this case $\mathbf{b}$ must be outside the grey area.

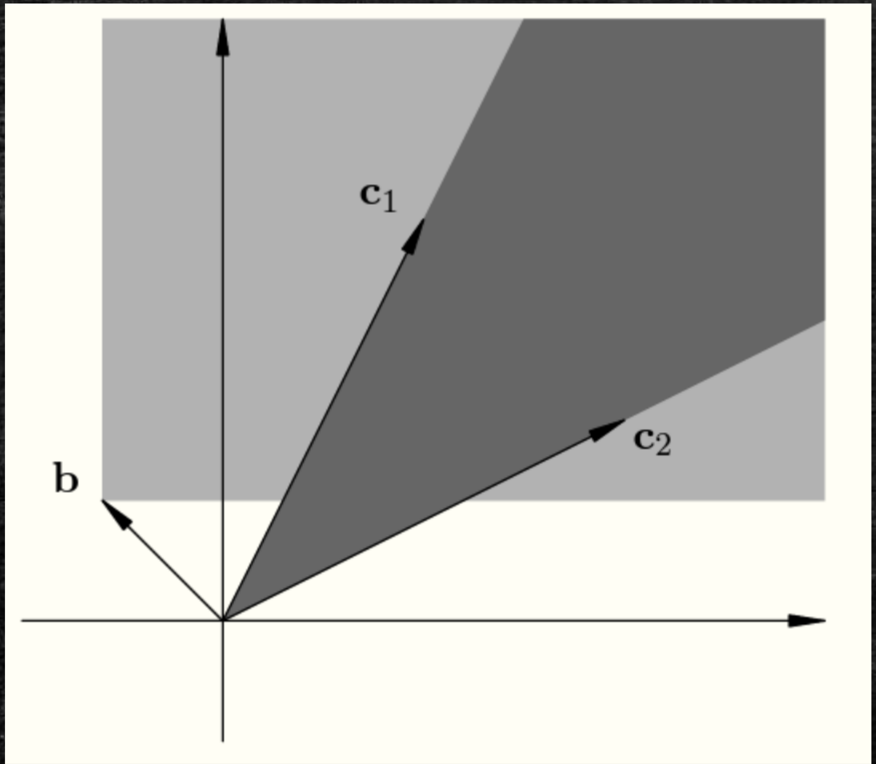
Illustration for $A = [\boldsymbol{c}_1 \quad \boldsymbol{c}_2]$



A Corollary to Farkas Lemma

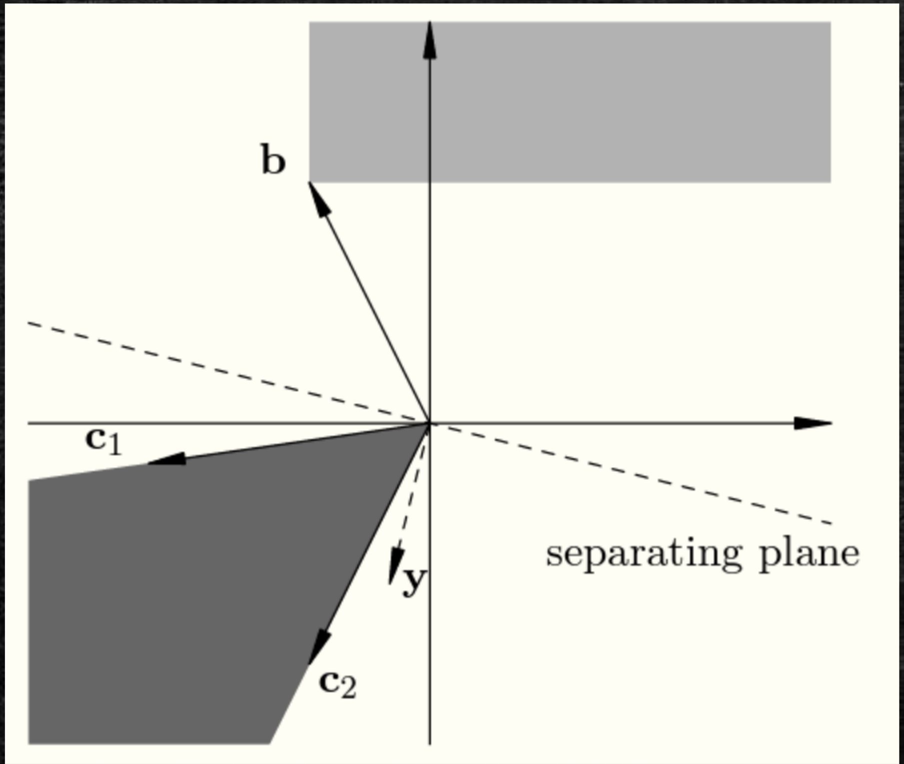
- **Corollary**. Exactly one of the followings holds for matrix $A \in \mathbb{R}^{m \times n}$ and vector $\mathbf{b} \in \mathbb{R}^m$:
  1. There exists $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \geq \mathbf{0}$ such that $A\mathbf{x} \geq \mathbf{b}$.
  2. There exists $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y} \leq \mathbf{0}$ such that $A^\top \mathbf{y} \geq \mathbf{0}$ and $\boldsymbol{b}^\top \mathbf{y} < 0$.

Case 1 of the Corollary



- $\{A\mathbf{x} \mid \mathbf{x} \geq \mathbf{0}\}$ is the dark grey area.
- $\{\mathbf{x} \mid \mathbf{x} \geq \mathbf{b}\}$ is the light grey area.
- 1 says that the two areas intersect.

Case 2 of the Corollary



- $\{A\mathbf{x} \mid \mathbf{x} \geq \mathbf{0}\}$ is the dark grey area.
- $\{\mathbf{x} \mid \mathbf{x} \geq \mathbf{b}\}$ is the light grey area.
- 2 describes that the two areas do not intersect.
- We can find a separating plane with normal vector $\mathbf{y}$.
  - Thus, $A^\top \mathbf{y} \geq 0$ and $\mathbf{b}^\top \mathbf{y} < 0$
- We must have $\mathbf{y} \leq \mathbf{0}$:
  - If this fails for one entry: $y_i > 0$
  - $\mathbf{z} = (\varepsilon, \dots, \varepsilon, z_i = 1, \varepsilon, \dots, \varepsilon)$ and $\boldsymbol{y}$ on same side
  - $\mathbf{z}$ is in the first quadrant, and it will eventually intersect the light grey area after extension.
  - The two areas are on the same side with $\mathbf{y}$.

Proof of the Corollary

- Define $A' \in \mathbb{R}^{m\times(n+m)}$ by $A' = [A \quad -I]$.
- Apply Farkas Lemma on A' and $\mathbf{b}$.
- Let P1 and P2 be 1 and 2 in Farkas Lemma; Q1 and Q2 be 1 and 2 in the corollary.
- We aim to show P1 $\Leftrightarrow$ P2 and Q1 $\Leftrightarrow$ Q2.

Proof of the Corollary

- Define $A' \in \mathbb{R}^{m\times(n+m)}$ by $A' = [A \quad -I]$.
- P1 $\Leftrightarrow \exists \mathbf{x}' \in \mathbb{R}^{n+m}$ s.t. $\mathbf{x}' \geq \mathbf{0}$ and $A'\mathbf{x}' = \mathbf{b}$.
- (by writing $\mathbf{x}' = \begin{bmatrix}\mathbf{x}\\ \bar{\mathbf{x}}\end{bmatrix}$) $\Leftrightarrow$ $[A \quad -I]\begin{bmatrix}\mathbf{x}\\ \bar{\mathbf{x}}\end{bmatrix} = \mathbf{b}$ (where $\mathbf{x} \geq \mathbf{0}$, $\bar{\mathbf{x}} \geq \mathbf{0}$)
- $\Leftrightarrow$ $A\mathbf{x} - \bar{\mathbf{x}} = \mathbf{b}$ $\Leftrightarrow A\mathbf{x} \geq \mathbf{b}$ (since $\bar{\mathbf{x}} \geq \mathbf{0}$)
- $\Leftrightarrow$ Q1

Proof of the Corollary

- Define $A' \in \mathbb{R}^{m\times(n+m)}$ by $A' = [A \quad -I]$.
- P2 $\Leftrightarrow \exists \mathbf{y} \in \mathbb{R}^m$ s.t. $A'^{\top}\mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}^{\top}\mathbf{y} < 0$.
- $\Leftrightarrow \begin{bmatrix} A^{\top} \\ -I \end{bmatrix}\mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}^{\top}\mathbf{y} < 0$
- $\Leftrightarrow A^{\top}\mathbf{y} \geq \mathbf{0}$, $-\mathbf{y} \geq 0$, and $\mathbf{b}^{\top}\mathbf{y} < 0$
- $\Leftrightarrow$ Q2

Now we are ready to prove strong duality theorem…

- Weak duality: $\mathbf{c}^\top \mathbf{x} \leq \mathbf{b}^\top \mathbf{y}^*$ holds for any $\mathbf{x} \geq \mathbf{0}$.
- Suppose strong duality fails: $\mathbf{c}^\top \mathbf{x} < \mathbf{b}^\top \mathbf{y}^*$.
- There does not exist $\mathbf{x} \geq \mathbf{0}$ satisfying $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{c}^\top \mathbf{x} \geq \mathbf{b}^\top \mathbf{y}^*$.
- We cannot have $\begin{bmatrix} -A \\ \mathbf{c}^\top \end{bmatrix} \mathbf{x} \geq \begin{bmatrix} -\mathbf{b} \\ \mathbf{b}^\top \mathbf{y}^* \end{bmatrix}$ and $\mathbf{x} \geq \mathbf{0}$.
- Q1 in corollary fails for matrix $\begin{bmatrix} -A \\ \mathbf{c}^\top \end{bmatrix}$ and vector $\begin{bmatrix} -\mathbf{b} \\ \mathbf{b}^\top \mathbf{y}^* \end{bmatrix}$.
- Thus, Q2 must be true.

Now we are ready to prove strong duality theorem…

- Q2 is true for matrix $\begin{bmatrix} -A \\ \mathbf{c}^\top \end{bmatrix}$ and vector $\begin{bmatrix} -\mathbf{b} \\ \mathbf{b}^\top \mathbf{y}^* \end{bmatrix}$.
- There exist $\mathbf{y} \in \mathbb{R}^m$ and $w \in \mathbb{R}$ such that

$$[-A^\top \quad \mathbf{c}] \begin{bmatrix} \mathbf{y} \\ w \end{bmatrix} \geq \mathbf{0}, \quad [-\mathbf{b}^\top \quad \mathbf{b}^\top \mathbf{y}^*] \begin{bmatrix} \mathbf{y} \\ w \end{bmatrix} < 0, \quad \text{and} \quad \begin{bmatrix} \mathbf{y} \\ w \end{bmatrix} \leq \mathbf{0}.$$

- After matrix multiplications,

$$\begin{cases} -A^\top \mathbf{y} + w\mathbf{c} \geq \mathbf{0} \\ -\mathbf{b}^\top \mathbf{y} + w\mathbf{b}^\top \mathbf{y}^* < 0 \\ \mathbf{y} \leq \mathbf{0} \\ w \leq 0 \end{cases}$$

Proof of Strong Duality Theorem

$$\begin{cases} -A^{\top}\mathbf{y} + w\boldsymbol{c} \geq \mathbf{0} \\ -\mathbf{b}^{\top}\mathbf{y} + w\mathbf{b}^{\top}\mathbf{y}^{*} < 0 \\ \mathbf{y} \leq \mathbf{0} \\ w \leq 0 \end{cases}$$

- Suppose $w < 0$. We divide w on both sides:

$$\begin{cases} -A^{\top}\left(\dfrac{\mathbf{y}}{w}\right) + \mathbf{c} \leq \mathbf{0} \\ -\mathbf{b}^{\top}\left(\dfrac{\mathbf{y}}{w}\right) + \mathbf{b}^{\top}\mathbf{y}^{*} > 0 \\ \left(\dfrac{\mathbf{y}}{w}\right) \geq \mathbf{0} \end{cases}$$

- $\left(\frac{\mathbf{y}}{w}\right)$ is a better solution than $\mathbf{y}^{*}$ in the dual LP, contradiction!

Proof of Strong Duality Theorem

$$\begin{cases} -A^\top \mathbf{y} + w\mathbf{c} \geq \mathbf{0} \\ -\boldsymbol{b}^\top \mathbf{y} + w\mathbf{b}^\top \mathbf{y}^* < 0 \\ \mathbf{y} \leq \mathbf{0} \\ w \leq 0 \end{cases}$$

- Let's then do the case $w = 0$.
- We have $-A^\top \mathbf{y} \geq \mathbf{0}$, $-\mathbf{b}^\top \mathbf{y} < 0$, and $\mathbf{y} \leq \mathbf{0}$.
- Q2 in Corollary holds for $-A$ and $-\mathbf{b}$.
- So Q1 must be false: $\nexists \mathbf{x} \geq 0: (-A)\mathbf{x} \geq -\mathbf{b}$.
- The feasible region for the primal LP is empty!

Part III: LP-Relaxation

Integer Program

- If we require each variable in a linear program is an integer, we obtain an integer program (IP), or integer linear program (ILP).
- Many problem can be formulated as IP.
- Standard form:

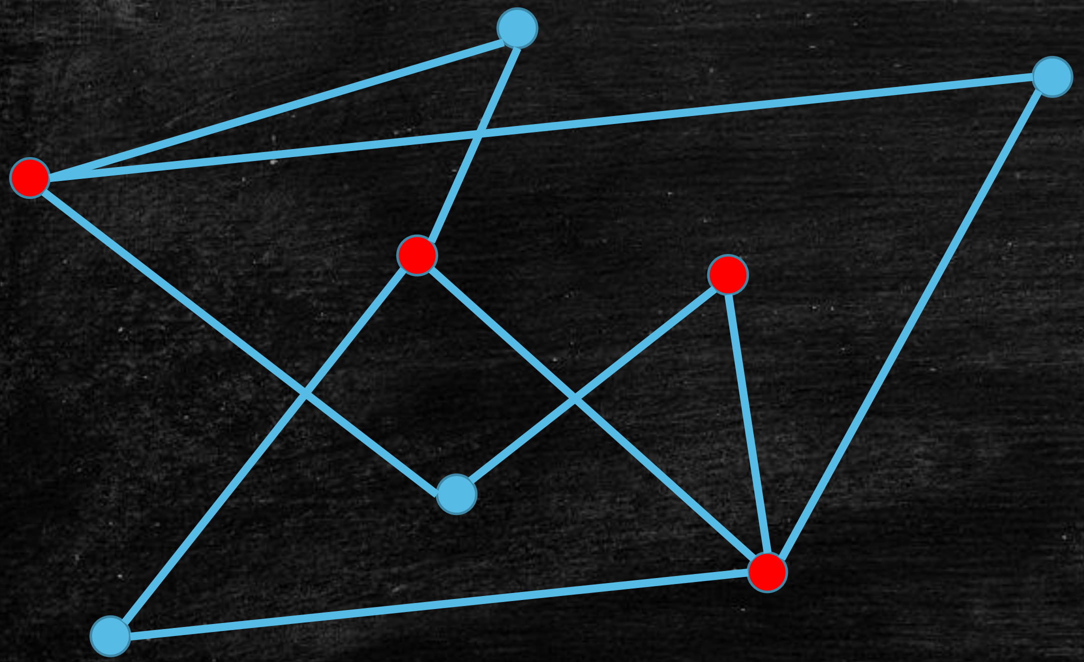
$$
\begin{aligned}
\text{maximize} \quad & \mathbf{c}^\top \mathbf{x} \\
\text{subject to} \quad & A\mathbf{x} \leq \mathbf{b} \\
& \mathbf{x} \geq \mathbf{0} \\
& \mathbf{x} \in \mathbb{Z}^n
\end{aligned}
$$

LP-Relaxation

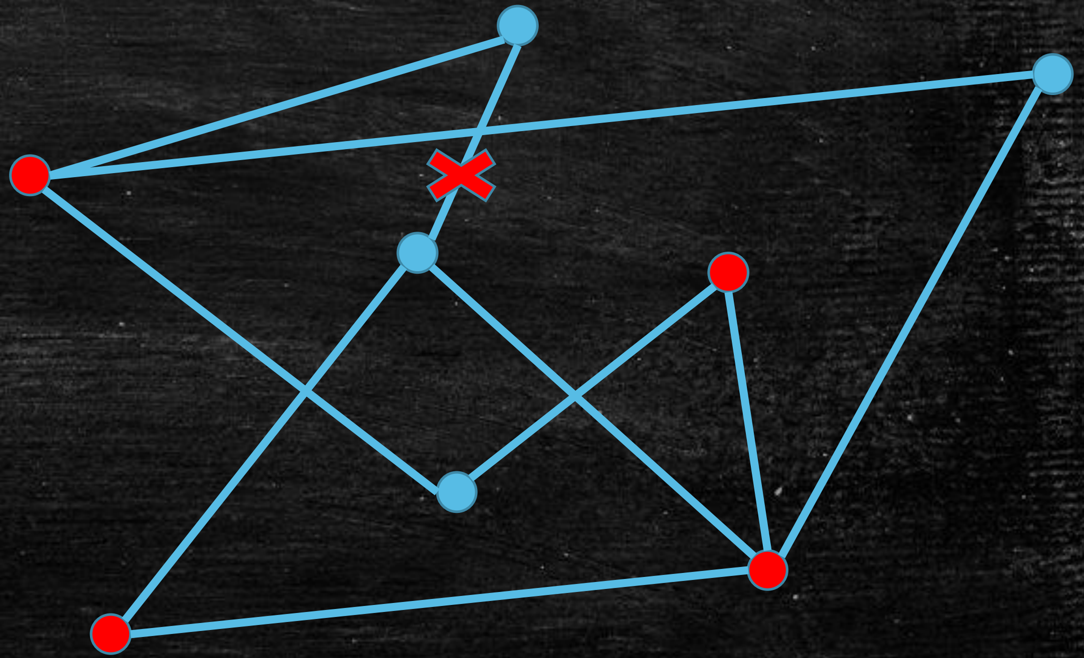
- Integer Programming is NP-complete, even for the zero-one special case $\forall i$: $x_i \in \{0, 1\}$.
- We can use the fact that LP is polynomial-time solvable to design approximation algorithm.
- Relax $x_i \in \{0,1\}$ to $0 \leq x_i \leq 1$.
- Then "round" the fractional solution to integral one:
  - E.g., $x_i = 0.7$ is rounded to $x_i = 1$, $x_i = 0.2$ is rounded to $x_i = 0$.
- and show that the rounded solution is feasible and achieves good approximation guarantee.

LP-Relaxation Example: Vertex Cover

- Given an undirected graph $G = (V, E)$, a subset of vertices $S \subseteq V$ is a vertex cover if S contains at least one endpoint of every vertex.



a vertex cover



not a vertex cover

LP-Relaxation Example: Vertex Cover

Problem [(Minimum) Vertex Cover]. Given an undirected graph, find a vertex cover with minimum number of vertices.

- Formulation by integer program:
  - $x_u = 1$ represents $u \in V$ is selected in the cover; $x_u = 0$ otherwise.

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{v \in V} x_v & \\
\text{subject to} & x_u + x_v \geq 1 & \forall (u, v) \in E \\
 & x_v \in \{0, 1\} & \forall v \in V
\end{array}
$$

LP-Relaxation Example: Vertex Cover

Problem [(Minimum) Vertex Cover]. Given an undirected graph, find a vertex cover with minimum number of vertices.

- Relax it to a linear program below:

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{v \in V} x_v & \\
\text{subject to} & x_u + x_v \geq 1 & \forall (u, v) \in E \\
 & 0 \leq x_v \leq 1 & \forall v \in V
\end{array}
$$

LP-Relaxation Example: Vertex Cover

- OPT(IP) – optimal objective value $\sum_{v\in V} x_v$ for IP
  - This is the objective we want for vertex cover
- OPT(LP) – optimal objective value $\sum_{v\in V} x_v$ for LP
- OPT(IP) $\geq$ OPT(LP): because LP has a larger feasible region.

$$
\begin{aligned}
\text{minimize} \quad & \sum_{v\in V} x_v \\
\text{subject to} \quad & x_u + x_v \geq 1 \qquad \forall (u,v) \in E \\
& x_v \in \{0,1\} \qquad \forall v \in V
\end{aligned}
$$

Integer Program (IP)

$$
\begin{aligned}
\text{minimize} \quad & \sum_{v\in V} x_v \\
\text{subject to} \quad & x_u + x_v \geq 1 \qquad \forall (u,v) \in E \\
& 0 \leq x_v \leq 1 \qquad \forall v \in V
\end{aligned}
$$

Linear Program (LP)

LP-Relaxation Example: Vertex Cover

An approximation algorithm for vertex cover:

- Formulate the problem as an integer program and obtain its LP-relaxation.
- Solve the linear program and obtain its optimal solution $\{x_v^*\}_{v\in V}$.
- Return $S = \{v \mid x_v^* \geq \frac{1}{2}\}$

Correctness

S returned by the algorithm is vertex cover.

- Proof. Consider an arbitrary edge $(u, v) \in E$.
- We have $x_u^* + x_v^* \geq 1$ by feasibility, which implies we have either $x_u^* \geq \frac{1}{2}$ or $x_v^* \geq \frac{1}{2}$, or both.
- By our algorithm, we have either $u \in S$ or $v \in S$, or both.

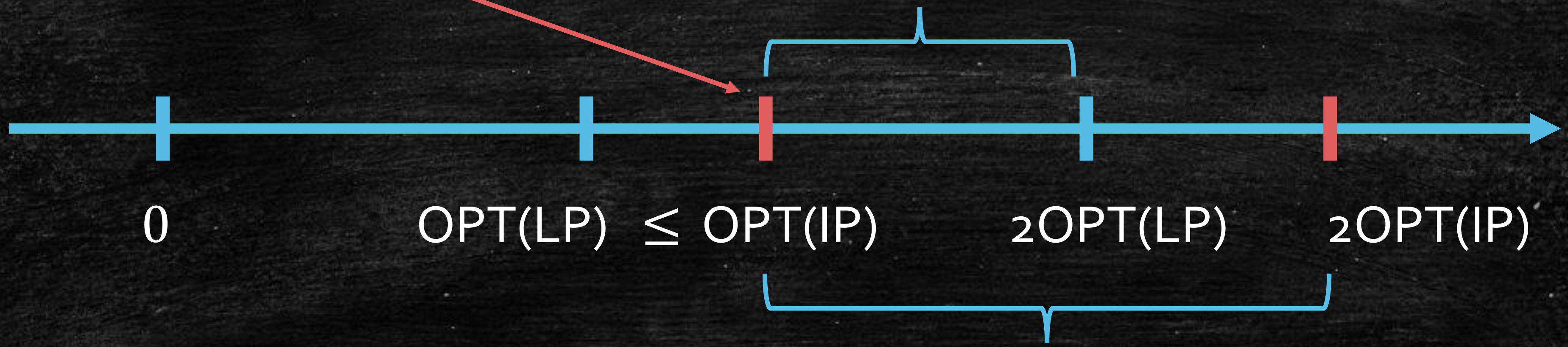
The algorithm is a 2-approximation.

The algorithm is a 2-approximation algorithm: $|S| \leq 2 \cdot \mathrm{OPT(IP)}$.

- Proof. Since we have $\mathrm{OPT(IP)} \geq \mathrm{OPT(LP)}$, it suffices to prove $|S| \leq 2 \cdot \mathrm{OPT(LP)}$.

The optimal solution
for vertex cover

We will prove $|S|$ is within here.



To show 2-approximation, $|S|$ is required to be within here.

The algorithm is a 2-approximation.

The algorithm is a 2-approximation algorithm: $|S| \leq 2 \cdot \mathrm{OPT(IP)}$.

- Proof. Since we have $\mathrm{OPT(IP)} \geq \mathrm{OPT(LP)}$, it suffices to prove $|S| \leq 2 \cdot \mathrm{OPT(LP)}$.
- $\mathrm{OPT(LP)} = \sum_{v \in V} x_v^* = \sum_{v : x_v^* < \frac{1}{2}} x_v^* + \sum_{v : x_v^* \geq \frac{1}{2}} x_v^*$
- $\geq \sum_{v : x_v^* < \frac{1}{2}} 0 + \sum_{v : x_v^* \geq \frac{1}{2}} \frac{1}{2} = \frac{1}{2} \cdot |S|$
- which implies $|S| \leq 2 \cdot \mathrm{OPT(LP)}$.

Today’s Lecture

- Introduction to Linear Programming
- LP Duality Theorem
- LP-Relaxation – use LP to design approximation algorithms