Applications of LP-Duality

Max-Flow-Min-Cut Theorem Revisit, von Neumann’s Minimax Theorem

Strong Duality Theorem

- Theorem [Strong Duality Theorem]. Let $\mathbf{x}^*$ be the optimal solution to (a) and $\mathbf{y}^*$ be the optimal solution to (b), then $\mathbf{c}^\top \mathbf{x}^* = \mathbf{b}^\top \mathbf{y}^*$.

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge \mathbf{0} \end{aligned} \qquad \text{(a)}$$

$$\begin{aligned} \text{minimize} \quad & \mathbf{b}^\top \mathbf{y} \\ \text{subject to} \quad & A^\top \mathbf{y} \ge \mathbf{c} \\ & \mathbf{y} \ge \mathbf{0} \end{aligned} \qquad \text{(b)}$$

Primal feasible | Primal OPT = Dual OPT | Dual feasible

Part I: Max-Flow-Min-Cut Theorem Revisited

Strong LP-Duality $\Longrightarrow$ Max-Flow-Min-Cut

Use Strong Duality Theorem to prove max-flow-min-cut theorem:

- Step 1: Write down the LP for max-flow problem.
- Step 2: Show that the dual program describes the fractional version of the min-cut problem.
- Step 3: Show that the dual program always have integral optimum.
  - So that the dual optimum is indeed the size of min-cut.
- Step 4: apply Strong Duality Theorem to show max-flow = min-cut

The Maximum Flow Problem

- The maximum flow problem can be formulated by a linear program.

$$\begin{aligned}
&\text{maximize} && \sum_{u:(s,u)\in E} f_{su} && \\
&\text{subject to} && 0 \le f_{uv} \le c_{uv} && \forall (u,v) \in E \\
& && \sum_{v:(v,u)\in E} f_{vu} = \sum_{w:(v,w)\in E} f_{uw} && \forall u \in V \setminus \{s,t\}
\end{aligned}$$

Let’s Write It in Standard Form

$$
\begin{array}{lll}
\text{maximize} & \displaystyle\sum_{u:(s,u)\in E} f_{su} & \\
\text{subject to} & f_{uv} \le c_{uv} & \forall (u,v) \in E \\
& \displaystyle\sum_{v:(v,u)\in E} f_{vu} - \sum_{w:(u,w)\in E} f_{uw} \le 0 & \forall u \in V \setminus \{s,t\} \\
& \displaystyle -\sum_{v:(v,u)\in E} f_{vu} + \sum_{w:(u,w)\in E} f_{uw} \le 0 & \forall u \in V \setminus \{s,t\} \\
& f_{uv} \ge 0 & \forall (u,v) \in E
\end{array}
$$

Compute Its Dual Program

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{(u,v)\in E} c_{uv} y_{uv} & \\
\text{subject to} & y_{su} + z_u \geq 1 & \forall u: (s,u) \in E \\
& y_{vt} - z_v \geq 0 & \forall v: (v,t) \in E \\
& y_{uv} - z_u + z_v \geq 0 & \forall (u,v) \in E, u \neq s, v \neq t \\
& y_{uv} \geq 0 & \forall (u,v) \in E
\end{array}
$$

- We aim to show the LP above describes the min-cut problem.
- Let $\mathrm{OPT}_{\mathrm{dual}}$ be its optimal objective value. We need to show $\mathrm{OPT}_{\mathrm{dual}}$ is the size of the min-cut.

Some Intuitions

$$
\begin{aligned}
&\text{minimize} && \sum_{(u,v)\in E} c_{uv} y_{uv} && \\
&\text{subject to} && y_{su} + z_u \geq 1 && \forall u\colon (s,u) \in E \\
& && y_{vt} - z_v \geq 0 && \forall v\colon (v,t) \in E \\
& && y_{uv} - z_u + z_v \geq 0 && \forall (u,v) \in E, u \neq s, v \neq t \\
& && y_{uv} \geq 0 && \forall (u,v) \in E
\end{aligned}
$$

- y_{uv} describes if edge (u,v) is cut:

$$y_{uv} = \begin{cases} 1 & \text{if } (u,v) \text{ is cut} \\ 0 & \text{otherwise} \end{cases}$$

- z_u describes u's "side":

$$z_u = \begin{cases} 1 & \text{if } u \text{ is on the } s-\text{side} \\ 0 & \text{if } u \text{ is on the } t-\text{side} \end{cases}$$

Turn Intuitions to Formal Proof

- y_{uv} describes if edge (u, v) is cut:

$$y_{uv} = \begin{cases} 1 & \text{if } (u, v) \text{ is cut} \\ 0 & \text{otherwise} \end{cases}$$

- z_u describes u's "side":

$$z_u = \begin{cases} 1 & \text{if } u \text{ is on the } s-\text{side} \\ 0 & \text{if } u \text{ is on the } t-\text{side} \end{cases}$$

To turn our intuitions to a formal proof, we will show

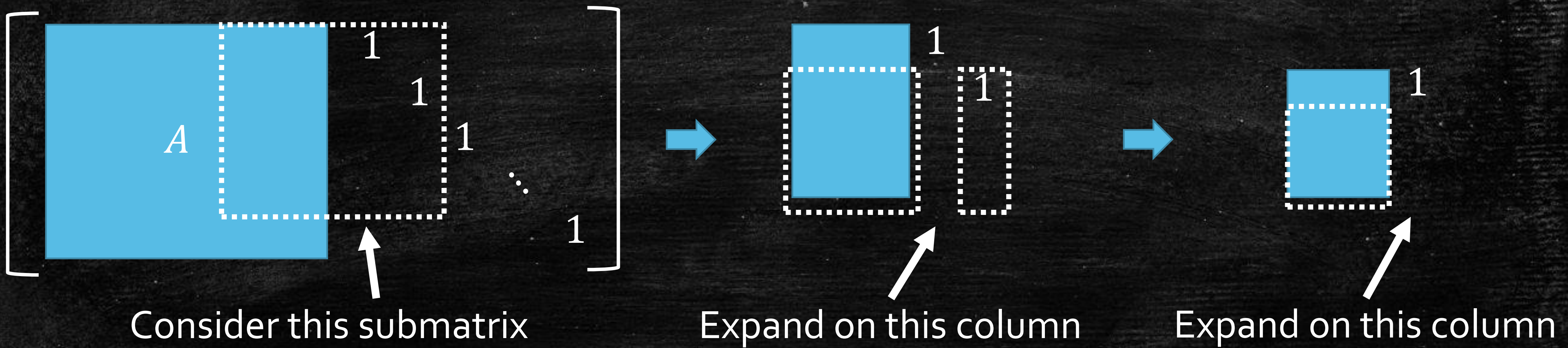
- There is an optimal solution with $y_{uv}, z_u \in \mathbb{Z}$,
  - A common method: total unimodularity
- and furthermore, there is an optimal solution with $y_{uv} \in \{0, 1\}$.
  - If $y_{uv} \geq 2$ for some $(u, v) \in E$, then the solution cannot be optimal.
- The optimal integral solution exactly gives a min-cut.

Totally Unimodular Matrix

- **Definition.** A matrix A is totally unimodular if every square submatrix has determinant 0, 1 or -1.
- **Theorem.** If $A \in \mathbb{R}^{m \times n}$ is totally unimodular and $\mathbf{b}$ is an integer vector, then the polytope $P = \{\mathbf{x}: A\mathbf{x} \leq \mathbf{b}\}$ has integer vertices.
- Proof. If $\mathbf{v} \in \mathbb{R}^n$ is a vertex of P. Then there exists an invertible square submatrix A' of A such that $A'\mathbf{v} = \mathbf{b}'$ for some sub-vector $\mathbf{b}'$ of $\mathbf{b}$.
- By Cramer's Rule, we have $v_i = \frac{\det(A_i'|\mathbf{b}')}{\det(A_i')}$, where $(A_i'|\mathbf{b}')$ is the matrix with i-th column replaced by $\mathbf{b}'$.
- $\det(A_i') = \pm 1$ and $\det(A_i'|\mathbf{b}') \in \mathbb{Z}$. Thus, $\mathbf{v}$ is integral.

Some Simple Observations

- If A is totally unimodular, then so are $A^\top, [I \;\; A], [A \;\; I], \begin{bmatrix} I \\ A \end{bmatrix}$, and $\begin{bmatrix} A \\ I \end{bmatrix}$. If any of $A^\top, [I \;\; A], [A \;\; I], \begin{bmatrix} I \\ A \end{bmatrix}$, and $\begin{bmatrix} A \\ I \end{bmatrix}$ is totally unimodular, then so is A.
- Proof. Just expand the determinant and you will see it…
- The determinant of $[A \;\; I]$ equals to ± 1 times the determinant of some square submatrix of A.



Corollary on Integrality of LP

- **Theorem.** If $A \in \mathbb{R}^{m \times n}$ is totally unimodular and $\mathbf{b}$ is an integer vector, then the polytope $P = \{\mathbf{x}: A\mathbf{x} \leq \mathbf{b}\}$ has integer vertices.
- Since there always exists optimum at a vertex of the feasible region of LP, we have the following corollary.
- **Corollary.** If A is unimodular, then the optimal solution to LP (a) is integral when $\mathbf{b}$ is integral, and the optimal solution to LP (b) is integral when $\mathbf{c}$ is integral.

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \qquad \text{(a)}$$

$$\begin{aligned} \text{minimize} \quad & \mathbf{b}^\top \mathbf{y} \\ \text{subject to} \quad & A^\top \mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} \end{aligned} \qquad \text{(b)}$$

Proving Integrality of y_{uv}, z_u

$$
\begin{array}{lll}
\text{minimize} & \sum_{(u,v)\in E} c_{uv} y_{uv} & \\
\text{subject to} & y_{su} + z_u \geq 1 & \forall u: (s,u) \in E \\
 & y_{vt} - z_v \geq 0 & \forall v: (v,t) \in E \\
 & y_{uv} - z_u + z_v \geq 0 & \forall (u,v) \in E, u \neq s, v \neq t \\
 & y_{uv} \geq 0 & \forall (u,v) \in E
\end{array}
$$

- Now, we show that the matrix describing the first three rows of the constraints is totally unimodular.

Proving Integrality of y_{uv}, z_u

- The matrix can be written below:

$$
\begin{array}{cccccc}
 & \overbrace{\qquad\qquad\qquad}^{|E|} & \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad\qquad\qquad}^{|V|}} & \\
 & & s & u & v & t & \\
|E|\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. & \left[\begin{array}{c} \\ |E|\times|E| \\ \text{identity} \\ \text{matrix} \\ \\ \end{array}\right. & \begin{array}{c} 0 \\ \\ \\ \\ \\ \end{array} & \begin{array}{c} 1 \\ \\ -1 \\ \\ \\ \end{array} & \begin{array}{c} \\ \\ 1 \\ \\ -1 \end{array} & \left.\begin{array}{c} \\ \\ \\ \\ 0 \end{array}\right] & \begin{array}{c} (s,u) \\ \\ (u,v) \\ \\ (v,t) \end{array} \\
 & Y & \multicolumn{4}{c}{Z} &
\end{array}
$$

- Let the matrix be $[Y \;\; Z]$. Y is the identity matrix. We only need to show Z is totally unimodular.

Proving Z is totally unimodular by Induction...

- Base Step: Each cell of Z belongs to $\{0, 1, -1\}$.
- Inductive Step: Suppose every $k \times k$ submatrix of Z has determinant belongs to $\{0, 1, -1\}$. Consider any $(k+1) \times (k+1)$ submatrix Z'.
- Case 1: If a row of Z' is all-zero, then $\det(Z') = 0$.
- Case 2: If a row of Z' contains only one non-zero entry, then $\det(Z')$ equals to ± 1 times the determinant of a $k \times k$ submatrix. $\det(Z') \in \{0, 1, -1\}$ by induction hypothesis.
- Case 3: If every row of Z' has two non-zero entries (one of them is -1 and the other is 1), then $\det(Z') = 0$:
  - Adding all the column vectors, we get a zero vector.

Proving Integrality of y_{uv}, z_u

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{(u,v)\in E} c_{uv} y_{uv} & \\
\text{subject to} & y_{su} + z_u \geq 1 & \forall u: (s,u) \in E \\
& y_{vt} - z_v \geq 0 & \forall v: (v,t) \in E \\
& y_{uv} - z_u + z_v \geq 0 & \forall (u,v) \in E, u \neq s, v \neq t \\
& y_{uv} \geq 0 & \forall (u,v) \in E
\end{array}
$$

- Now, we conclude that there exists an optimal solution with $y_{uv}, z_u \in \mathbb{Z}$.

Some Intuitions

- Consider an arbitrary s-t path $s - v_1 - v_2 - \cdots - v_{\ell-1} - t$.
- Sum up all the constraints for the edges on the path:

$$\left(y_{sv_1} + z_{v_1}\right) + \left(y_{v_{\ell-1}t} - z_{v_{\ell-1}}\right) + \sum_{i=1}^{\ell-2}\left(y_{u_iu_{i+1}} - z_{u_i} + z_{u_{i+1}}\right) \geq 1$$

$$\Longrightarrow \quad y_{sv_1} + y_{v_{\ell-1}t} + \sum_{i=1}^{\ell-2} y_{u_iu_{i+1}} \geq 1$$

- Conclusion: We must have $y_{uv} \geq 1$ for at least one edge (u, v) on the path.
- Removing $\{(u, v): y_{uv} \geq 1\}$ disconnects t from s.

$\mathrm{OPT}_{\mathrm{dual}}$ is an upper-bound to min-cut.

- **Lemma 1.** $\mathrm{OPT}_{\mathrm{dual}}$ is an upper-bound to min-cut.
- Proof. Let $(\mathbf{y}^*, \mathbf{z}^*)$ be an integral optimal solution.
- Let $C = \{(u, v) \in E : y_{uv}^* \geq 1\}$. We have shown removing C disconnect t from s.
- Let $L \subseteq V$ be the vertices reachable from s after removing C, and $R = V \setminus L$. Then $\{L, R\}$ is an s-t cut.
- For min-cut $\{L^*, R^*\}$, we have

$$c(L^*, R^*) \leq c(L, R) = \sum_{(u,v)\in E: u\in L, v\in R} c_{uv} \leq \sum_{(u,v)\in E: u\in L, v\in R} c_{uv} y_{uv}^* = \mathrm{OPT}_{\mathrm{dual}}$$

$\mathrm{OPT}_{\mathrm{dual}}$ is also a lower-bound to min-cut.

- **Lemma 2.** $\mathrm{OPT}_{\mathrm{dual}}$ is a lower-bound to min-cut.
- Proof. Let $\{L^*, R^*\}$ be a min-cut. We construct a LP solution:
- $y_{uv} = \begin{cases} 1 & \text{if } u \in L^*, v \in R^* \\ 0 & \text{otherwise} \end{cases}$ and $z_u = \begin{cases} 1 & \text{if } u \in L^* \\ 0 & \text{if } u \in R^* \end{cases}$
- It is easy to verify that the solution is feasible…
- Then,

$$\mathrm{OPT}_{\mathrm{dual}} \leq \sum_{(u,v)\in E} c_{uv} y_{uv} = \sum_{(u,v)\in E: u\in L^*, v\in R^*} c_{uv} = c(L^*, R^*)$$

Now we conclude Max-Flow-Min-Cut Theorem

- By the two lemmas, $\mathrm{OPT}_{\mathrm{dual}}$ equals to the size of min-cut.
- By the strong duality theorem, $\mathrm{OPT}_{\mathrm{dual}}$ equals to the size of max-flow.
- Thus, the size of min-cut equals the size of max-flow.

A Framework for Proving Theorems Using Strong Duality

- Write down the primal and dual LPs.
- Justify that the primal and dual LPs describe the corresponding problems.
- If the problem described is discrete, prove that the corresponding LP always gives integral solution.
  - Total Unimodularity
- Apply strong duality theorem.

Revisiting Integrality Theorem for Max-Flow

- **Theorem.** If the capacities are all integers, then there exists an integral maximum flow.
- We have seen that "A" in the LP is totally unimodular
  - For dual program, we have proved $A^{\top}$ is totally unimodular.
- If all c_{uv} are integers, then vector "$\mathbf{b}$" in the LP is integral, and the LP has an integral optimal solution.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{u:(s,u)\in E} f_{su} \\
\text{subject to} \quad & f_{uv} \leq c_{uv} \\
& \sum_{v:(v,u)\in E} f_{vu} - \sum_{w:(u,w)\in E} f_{uw} \leq 0 \\
& -\sum_{v:(v,u)\in E} f_{vu} + \sum_{w:(u,w)\in E} f_{uw} \leq 0 \\
& f_{uv} \geq 0
\end{aligned}
$$

Part II: von Neumann's Minimax Theorem

Zero-Sum Game

- Two players: A and B
- Each player has a set of actions that (s)he can play.
  - Set of actions A can play: $\mathbf{a} = \{a_1, a_2, \dots, a_m\}$
  - Set of actions B can play: $\mathbf{b} = \{b_1, b_2, \dots, b_n\}$
- For each pair of actions (a_i, b_j) that two players play, an utility is assigned to each player: $u_A(a_i, b_j), u_B(a_i, b_j)$.
- A game is a zero-sum game if $\forall x_i, y_j: u_A(a_i, b_j) + u_B(a_i, b_j) = 0$.
- Payoff Matrix $G \in \mathbb{R}^{m \times n}$, where $G_{i,j}$ is the utility gain for A, or the utility loss for B, when (a_i, b_j) is played.

Example

- The payoff matrix for the Rock-Scissors-Paper game:

		Player B		
		Rock	Scissors	Paper
Player A	Rock	0	1	-1
	Scissors	-1	0	1
	Paper	1	-1	0

Strategy

- Set of actions A can play: $\mathbf{a} = \{a_1, a_2, \dots, a_m\}$
- A strategy for A is a probability distribution of $\mathbf{x}$.
- A pure strategy specifies one of $a_1, a_2, \dots, a_m$ with probability 1.
  - In other words, a pure strategy is an action.
- Otherwise, it is a mixed strategy.
  - In other words, a mixed strategy specify at least two actions with non-zero probability.
- Fix A's strategy, the best response for B is the strategy that maximizes B's utility.

Rock-Scissors-Paper Example

- A plays $(R, S, P) = (1, 0, 0)$:
  - It is a pure strategy that always plays “rock”.
  - The best response for B is $(0, 0, 1)$, with utility 1.
- A plays $(R, S, P) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$:
  - It is a mixed strategy.
  - The best response for B is $(0, 0, 1)$, with expected utility $\frac{1}{2} \times 1 + \frac{1}{4} \times 0 + \frac{1}{4} \times 0 = \frac{1}{2}$.
- A plays $(R, S, P) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$:
  - It is a mixed strategy.
  - Any strategy for B, pure or mixed, is a best response, with expected utility 0.

Expected Utility

- Let $\mathbf{x} = \{x_1, \dots, x_m\}$ and $\mathbf{y} = \{y_1, \dots, y_n\}$ be the strategies played by the two players.
- The expected utility for Player A is
$$U_A(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\top G \mathbf{y} = \sum_{i,j} G_{i,j} x_i y_j$$
- The expected utility for Player B is
$$U_B(\mathbf{x}, \mathbf{y}) = -\mathbf{x}^\top G \mathbf{y} = -\sum_{i,j} G_{i,j} x_i y_j$$

Does it matter who chooses strategy first?

Rock-Scissors-Paper: $G = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

- Suppose A chooses a strategy first.
  - Given that B will always play the best response
  - The optimal strategy for A is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$
  - Expected utility for both players is 0
- Suppose B chooses a strategy first.
  - Similar analysis, expected utility for both players is 0
- Same outcome regardless who chooses strategy first.
- Does it always hold for any zero-sum game?
- Yes! This is von Neumann's Minimax Theorem.

Minimax Theorem

- Suppose A chooses strategy first. Knowing that B will play the best response, A will choose an optimal strategy $\mathbf{x}$ that maximizes his/her utility:

$$\underbrace{\max_{\mathbf{x}} \overbrace{\min_{\mathbf{y}} \sum_{i,j} G_{i,j} x_i y_j}^{\text{B plays the best response given A's strategy $\mathbf{x}$.}}}_{\text{Given B plays the best response, A choose a strategy maximizing the utility.}}$$

- Suppose B chooses strategy first. Similarly, the utility for A is

$$\min_{\mathbf{y}} \max_{\mathbf{x}} \sum_{i,j} G_{i,j} x_i y_j$$

Minimax Theorem

- Minimax Theorem:

$$\max_{\mathbf{x}} \min_{\mathbf{y}} \sum_{i,j} G_{i,j} x_i y_j = \min_{\mathbf{y}} \max_{\mathbf{x}} \sum_{i,j} G_{i,j} x_i y_j$$

- Who chooses strategy first doesn't matter!

Pure Strategy Best Response

- **Lemma.** Fix A's strategy $\mathbf{x} = \{x_1, \ldots, x_m\}$, there exists a best response for B that is a pure strategy.
- Proof. Let $\mathbf{y} = \{y_1, \ldots, y_n\}$ be B's strategy.
- The utility for B is given by

$$-y_1 \sum_{i=1}^{m} G_{i,1} x_i - y_2 \sum_{i=1}^{m} G_{i,2} x_i - \cdots - y_n \sum_{i=1}^{m} G_{i,n} x_i$$

- Clearly, this is maximized if we set $y_i = 1$ where y_i has smallest coefficient.

LP formulation

- The lemma implies

$$\max_{\mathbf{x}} \min_{\mathbf{y}} \sum_{i,j} G_{i,j} x_i y_j = \max_{\mathbf{x}} \min_{j=1,\ldots,n} \sum_{i} G_{i,j} x_i$$

- Let z be the utility for Player A. The following LP formulates the max-min expression:

$$\begin{aligned}
&\text{maximize} && z \\
&\text{subject to} && \textstyle\sum_i G_{i,j} x_i \geq z && \forall j = 1, \ldots, n \\
& && x_1 + \cdots + x_m = 1 \\
& && x_1, \ldots, x_m \geq 0
\end{aligned}$$

Standard Form…

$$
\begin{aligned}
\text{maximize} \quad & z^+ - z^- \\
\text{subject to} \quad & -\sum_i G_{i,j} x_i + z^+ - z^- \leq 0 \qquad \forall j = 1, \dots, n \\
& x_1 + \cdots + x_m \leq 1 \\
& -x_1 - \cdots - x_m \leq -1 \\
& x_1, \dots, x_m, z^+, z^- \geq 0
\end{aligned}
$$

It's dual program is…

$$
\begin{array}{lll}
\text{minimize} & w^+ - w^- & \\
\text{subject to} & -\sum_j G_{i,j} y_i + w^+ - w^- \geq 0 & \forall i = 1, \dots, m \\
& y_1 + \cdots + y_n \geq 1 & \\
& -y_1 - \cdots - y_n \geq -1 & \\
& y_1, \dots, y_n, w^+, w^- \geq 0 &
\end{array}
$$

Simplify it, we get…

$$\begin{aligned} \text{minimize} \quad & w \\ \text{subject to} \quad & \textstyle\sum_j G_{i,j} y_i \le w \qquad\qquad \forall i = 1, \ldots, m \\ & y_1 + \cdots + y_n = 1 \\ & y_1, \ldots, y_n \ge 0 \end{aligned}$$

- This is exactly

$$\min_{\mathbf{y}} \max_{\mathbf{x}} \sum_{i,j} G_{i,j} x_i y_j = \min_{\mathbf{y}} \max_{i=1,\ldots,m} \sum_{i,j} G_{i,j} y_j$$

- Strong duality theorem $\Rightarrow$ Minimax Theorem.