1
Let's Count!

Combinatorics is the art of counting. You may have learned the two basic principles of counting, the *addition principle* (a.k.a. the *rule of sum*) and the *multiplication principle* (a.k.a. the *rule of product*), in primary schools. Now, let's start with these primary school mathematics.

1.1 *The twelvefold way*

Our first toy is the *balls-and-urns* model, which is simple but powerful. In this model, there are n balls and m urns, and we would like to *count / enumerate* how many ways to put these balls into urns. However, these balls and urns may be *distinct* or *identical*, and there may be some restrictions on the numbers of balls in each urn, such as *at most* one ball in each urn, or *at least* one ball in each urn. Depending on the different cases of balls, urns and restrictions, we can classify the counting problem into 12 typical tasks: $\{\text{distinct balls}, \text{identical balls}\} \times \{\text{distinct urns}, \text{identical urns}\} \times \{\text{no restrictions}, \text{at least one ball per urn}, \text{at most one ball per urn}\}$.

This classification is called the *twelvefold way*. The idea of the classification is credited to Gian-Carlo Rota, and the name was suggested by Joel Spencer.

If you are familiar with the language of functions or mappings, the twelvefold way can be viewed as counting the number of mappings from a set or an indistinguishable set of size n, to a set or an indistinguishable set of size m, where the mappings are subject to one of the three following restrictions: unrestricted, injective, or surjective.

Figure 1.1: Some of you may have known the twelvefold way in competitive programming

Formulas for the different cases of the twelvefold way are summarized in the following table.

Table 1.1: The twelvefold way

n balls	m urns	# of balls in each urn		
		arbitrary	≤ 1	≥ 1
distinct	distinct	m^n	$(m)_n$	$m!\, \left\{ n \atop m \right\}$
identical	distinct	$\binom{n+m-1}{m-1}$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
distinct	identical	$\sum_{k=1}^{m} \left\{ n \atop k \right\}$	$[n \leq m]$	$\left\{ n \atop m \right\}$
identical	identical	$p_m(n+m)$	$[n \leq m]$	$p_m(n)$

Now we explain the details of each case.

1. **Distinct balls, distinct urns, no restrictions**: Each ball can be put into any of m urns, so there is m^n ways by the multiplication principle.

2. **Distinct balls, distinct urns, at most** 1 **ball in each urn**: The first ball can be put into any of m urns, and the second one can be put into $m-1$ urns except the urn containing the first ball, and so on. So the number of ways in this case is

$$(m)_n \triangleq m \times (m-1) \times (m-2) \times \cdots \times (m-n+1)\,.$$

3. **Identical balls, distinct urns, at most** 1 **ball in each urn**: We choose n unordered urns from all m urns and put one ball in each urn. The number of ways in this case is the *binomial coefficient* (a.k.a. *combination*, or *combinatorial number*)

$$\binom{m}{n} \triangleq \frac{m!}{n!\,(m-n)!} = \frac{(m)_n}{n!}\,.$$

The name comes from the *binomial theorem*:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k\,,$$

and it usually read as "n choose k".

4. **Distinct balls, identical urns, at most** 1 **ball in each urn**: Clearly if $n > m$ there is no solution. If $n \leq m$, since all urns are indistinguishable, there is a unique way to put balls into urns. We use notation $[n \leq m]$ to denote the indicator variable, that is,

$$[n \leq m] \triangleq \begin{cases} 1 & n \leq m \\ 0 & \text{otherwise} \end{cases}\,.$$

5. **Identical balls, identical urns, at most** 1 **ball in each urn**: The same as Case 4.

6. **Identical balls, distinct urns, at least 1 balls in each urn**: The number of ways in this case is equal to the number of solutions of $x_1 + x_2 + \cdots + x_m = n$ where $x_1, x_2, \ldots, x_m$ are positive integers. We can solve this task by the *stars-and-bars* technique. Suppose there are n stars in a line. Then we can place $m-1$ bars between the stars and let x_1 be the number of the first part of stars, x_2 be the number of the second part of stars, and so on. Because no urn is allowed to be empty (all the variables are positive), there is at most one bar between any pair of stars. For example, if $n = 6$ and $m = 3$, the following two bars give rise to the solution where $x_1 = 3$, $x_2 = 1$, and $x_3 = 2$.

$$\underbrace{\star\ \star\ \star}_{x_1} \mid \underbrace{\star}_{x_2} \mid \underbrace{\star\ \star}_{x_3}$$

The number of ways to put bars is the number of ways to choose $m-1$ positions from $n-1$ ones, that is,

$$\binom{n-1}{m-1}.$$

7. **Identical balls, distinct urns, no restrictions**: The number of ways in this case is equal to the number of solutions of $x_1 + x_2 + \cdots + x_m = n$ where $x_1, x_2, \ldots, x_m$ are nonnegative integers. Let $y_i = x_i + 1$ for all $i \in [m]$. Then it holds that $y_1 + y_2 + \cdots + y_n = m + n$ and all y_i's are positive integers. Because of the bijection between x_i and y_i, the number of nonnegative solutions of $x_1 + x_2 + \cdots + x_m = n$ is equal to the number of positive solutions of $y_1 + y_2 + \cdots + y_m = n + m$, which is, by Case 6,

$$\binom{n+m-1}{m-1}.$$

The number is also equal to the number of *multisets* of cardinality m, with elements taken from $[n]$. It is sometimes called the *multiset coefficient*, or *multiset number*.

Sometimes we use the notation

$$\left(\!\!\binom{m}{n}\!\!\right) = \binom{n+m-1}{n} = \binom{n+m-1}{m-1}.$$

8. **Distinct balls, identical urns, at least 1 ball in each urn**: The number of ways in this case is equal to the number of partitions of $[n]$ into m nonempty subsets. This number is called the *Stirling number of the second kind*, denoted by $S(n,m)$, or $\left\{ {n \atop m} \right\}$. Stirling numbers of the second kind obey the recurrence relation

$$\left\{ {n \atop m} \right\} = m \left\{ {n-1 \atop m} \right\} + \left\{ {n-1 \atop m-1} \right\} \qquad \text{for } 0 < m < n$$

with initial conditions

$$\left\{ {n \atop 0} \right\} = 0 \quad \text{for } n \geq 1 \quad \text{and} \quad \left\{ {n \atop n} \right\} = 1 \quad \text{for } n \geq 0\,.$$

We will revisit Stirling numbers and study its explicit formula of $\left\{ {n \atop m} \right\}$ in Sections 2.3 and 3.2.

Why?

9. **Distinct balls, distinct urns, at least 1 ball in each urn**: We first partition n distinct balls into m subsets, and then assign each urn with a subset. The number of partitions is $\left\{ {n \atop m} \right\}$ and the number of assignments is $m!$. So the number of ways in this case is $m! \left\{ {n \atop m} \right\}$.

10. **Distinct balls, identical urns, no restrictions**: We enumerate the number of nonempty urns. So the number of ways in this case is the sum of the number of ways to put n distinct balls into k nonempty urns over k.

If $m = n$, the number is called the *Bell number*:
$$B_n \triangleq \sum_{k=0}^{n} \left\{ {n \atop k} \right\}.$$

11. **Identical balls, identical urns, at least 1 ball in each urn**: The number of ways in this case is equal to the number of ways to represent n as a sum of m positive integers. We use $p_m(n)$ to denote this partition number, and $p_m(n)$ can be calculated using the recurrence relation $p_m(n) = p_{m-1}(n-1) + p_m(n-m)$ with the base case $p_1(n) = p_n(n) = 1$.

12. **Identical balls, identical urns, no restrictions**: The number of ways in this case is equal to the number of ways to represent n as a sum of m nonnegative integers. Similarly to Case 7, we can add 1 to each integer, so the number of ways in this case is equal to the number of ways to represent $n + m$ as a sum of m positive integers, i.e., $p_m(n + m)$.

1.2 *Binomial coefficients*

In this section, we introduce more on *binomial coefficients*.

Proposition 1.1. *Let n be a fixed number. Then*
$$\binom{n}{k} < \binom{n}{k+1}$$
if and only if $k + 1 \leq n/2$, and
$$\binom{n}{k} > \binom{n}{k+1}$$
if and only if $k \geq n/2$.

Proof.
$$\frac{\binom{n}{k}}{\binom{n}{k+1}} = \frac{(n)_k}{k!} \frac{(k+1)!}{(n)_{k+1}} = \frac{k+1}{n-k}.$$ □

We usually use the definition $\binom{n}{k} = \frac{(n)_k}{k!}$ instead of $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$. It will be helpful for genearal n ($n < k$ or $n \notin \mathbb{Z}$).

Proposition 1.1 states that the binomial coefficients is a *unimodal* sequence for any fixed n. In particular, it is further a *log-concave* sequence.

A sequence $\{a_k\}$ is said to be *concave* if it satisfies $2a_k \geq a_{k-1} + a_{k+1}$. A positive sequence $\{b_k\}$ is said to be *logarithmically concave*, or simply *log-concave*, if $\{\log b_k\}$ is concave.

Proposition 1.2.
$$\binom{n}{k}^2 > \binom{n}{k-1}\binom{n}{k+1}.$$

Proof.
$$\frac{\binom{n}{k-1}\binom{n}{k+1}}{\binom{n}{k}^2} = \frac{k(n-k)}{(k+1)(n-k+1)} < 1\,. \qquad \square$$

Proposition 1.3.
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n\,.$$

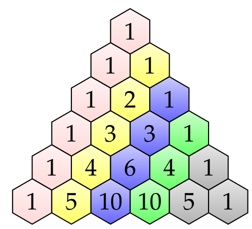
Proof. Note that $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. Let $x = 1$. $\square$

Proposition 1.4.
$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0\,.$$

Proof. Let $x = -1$ in $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. $\square$

The following proposition gives an important recurrence for binomial coefficients.

You may have known that binomial coefficients form the *Pascal's triangle*



Proposition 1.5.
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}\,.$$

Proof.
$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)_{k-1}}{(k-1)!} + \frac{(n-1)_k}{k!} \\ &= \frac{(n-1)_{k-1}}{k!} \cdot k + \frac{(n-1)_{k-1}}{k!} \cdot (n-k) \\ &= \frac{(n-1)_{k-1}}{k!} \cdot n = \binom{n}{k}\,. \end{aligned} \qquad \square$$

Sometimes the exact values of binomial coefficients are not necessary; instead, their approximate values or values modulo $\mathbb{Z}_p$ may suffice.

To evaluate the value of binomial coefficients asymptotically, a powerful tool is the **Stirling's approximation formula**:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n,$$

or more precisely,

$$\lim_{n\to\infty} \frac{n!\,e^n}{n^n\sqrt{n}} = \sqrt{2\pi}\,.$$

Roughly, the formula can most simply be derived by approximating the sum over the terms of the factorial with an integral, that is,

$$\begin{aligned}\ln n! = \sum_{k=1}^{n} \ln k &\approx \int_1^n \ln x \,\mathrm{d}x \\ &= n\ln n - n + 1\,.\end{aligned}$$

It yields the following simple but useful bound:

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{en}{k}\right)^k.$$

Further, a more precise approximation can be given by

$$\binom{n}{k} \approx \sqrt{\frac{n}{2\pi k(n-k)}}\,\frac{n^n}{k^k\,(n-k)^{n-k}} \geq \frac{1}{\sqrt{2\pi k}}\,\frac{n^n}{k^k\,(n-k)^{n-k}}\,,$$
$$\binom{n}{k} = \frac{(n)_k}{k!} \leq \frac{1}{\sqrt{2\pi k}}\,\frac{n\,(n-k/2)^{k-1}}{(k/e)^k} \leq \frac{1}{\sqrt{2\pi k}}\,\frac{n^k}{(k/e)^k}\,.$$

For even k, we have

$$(n)_k \leq n\left(n-\frac{k}{2}\right)^{k-1}.$$

For odd k, we have

$$(n)_k \leq \left(n-\frac{k-1}{2}\right)^{k}.$$

We may also concern the remainder of binomial coefficients divided by some prime p. For example, consider the following exercise.

Question 1.6. How many odd entries in the n-th row of the Pascal's triangle?

To answer this kind of questions, we introduce the *Lucas' theorem*.

Lucas (1878)

Theorem 1.7 (Lucas' Theorem)**.** *Let n, m be two nonnegative integers, and p be a prime. Suppose that m and n can be written as $n = \overline{n_k n_{k-1} \cdots n_0}$ and $m = \overline{m_k m_{k-1} \cdots m_0}$ in terms of base p, namely,*

$$n = n_k p^k + n_{k-1} p^{k-1} + \cdots + n_0 \quad \text{and}$$
$$m = m_k p^k + m_{k-1} p^{k-1} + \cdots + m_0\,,$$

where $n_k, n_{k-1}, \ldots, n_0, m_k, m_{k-1}, \ldots, m_0 \in \{0, 1, \ldots, p-1\}$. Then it holds that

$$\binom{n}{m} \equiv \prod_{i=0}^{k} \binom{n_i}{m_i} \pmod{p}\,.$$

Proof. It suffices to show that

$$\binom{n}{m} \equiv \binom{n \bmod p}{m \bmod p}\binom{\lfloor n/p \rfloor}{\lfloor m/p \rfloor} \pmod{p}\,.$$

For convenience we assume $n \geq m$ and $n, m \in \mathbb{N}$. Let $n = sp + r$ and $m = tp + w$ where $0 \leq r, w \leq p-1$. Note that

$$\begin{aligned} n! &= (sp+r)\cdots(sp+1)\prod_{i=0}^{s-1}\Big((ip+1)\cdots(ip+p)\Big) \\ &= (r! + \alpha_0 p)\cdot\prod_{i=1}^{s}\Big(ip\big((p-1)! + \alpha_i p\big)\Big) \\ &= s!\cdot p^s\cdot(r! + \alpha_0 p)\cdot\prod_{i=1}^{s}\big((p-1)! + \alpha_i p\big) \\ &= s!\cdot p^s\cdot\big((-1)^s r! + Ap\big) \end{aligned}$$

for some $\alpha_0, \ldots, \alpha_s, A \in \mathbb{N}$. Similarly, for $m = tp + w!$ we have

$$m! = t!\cdot p^t\cdot\big((-1)^t w! + Bp\big)$$

Here we apply the *Wilson's theorem*, which states that

$$(p-1)! \equiv -1 \pmod{p}\,.$$

for some $B \in \mathbb{N}$, and for $n - m = (s-t)p + (r-w)$, we have

$$(n-m)! = \begin{cases} (s-t)! \cdot p^{s-t} \cdot \big((-1)^{s-t}(r-w)! + Cp\big) & \text{if } r \geq w \\ (s-t-1)! \cdot p^{s-t-1} \cdot \big((-1)^{s-t-1}(r+p-w)! + Cp\big) & \text{if } r < w \end{cases}$$

for some $C \in \mathbb{N}$. Now, if $r \geq w$, it follows that

$$\begin{aligned}
\binom{n}{m} &\equiv \frac{n!}{m! \cdot (n-m)!} \\
&\equiv \frac{s! \cdot p^s \cdot \big((-1)^s r! + Ap\big)}{t! \cdot p^t \cdot \big((-1)^t w! + Bp\big) \cdot (s-t)! \cdot p^{s-t} \cdot \big((-1)^{s-t}(r-w)! + Cp\big)} \\
&\equiv \frac{s!}{t!(s-t)!} \cdot \frac{(-1)^s r! + Ap}{(-1)^s w!(r-w)! + Dp} \\
&\equiv \binom{s}{t}\binom{r}{w} \pmod{p}.
\end{aligned}$$

If $r < w$, we obtain $\binom{n}{m} \equiv 0 \pmod{p}$ since the numerator has term p^s but the denominator only has term p^{s-1}. But in this case we also have $\binom{r}{w} = 0$. Thus we conclude that $\binom{n}{m} \equiv \binom{s}{t}\binom{r}{w} \pmod{p}$. □

A generalized version of binomial coefficients is the *multinomial coefficients* in the *multinomial theorem*. Suppose $n = r_1 + \cdots + r_m$ where $r_1, \ldots, r_m$ are nonnegative integers. Define the multinomial coefficient as

$$\begin{aligned}
\binom{n}{r_1, \ldots, r_m} &\triangleq \binom{n}{r_1}\binom{n-r_1}{r_2} \cdots \binom{n - r_1 - \cdots - r_{m-1}}{r_m} \\
&= \frac{n!}{r_1! \, r_2! \, \cdots \, r_m!}.
\end{aligned}$$

Then the multinomial theorem states that

$$(x_1 + \cdots + x_m)^n = \sum_{r_1 + \cdots + r_m = n} \binom{n}{r_1, \ldots, r_m} x_1^{r_1} \cdots x_m^{r_m}.$$

1.3 *Combinatorial proofs and combinatorial identities*

In the last section, we prove some properties of binomial coefficients by explicit formulas. Now we introduce another type of proofs.

Alternative proof of Proposition 1.3. The right hand side counts the number of all subsets of $[n]$. The left hand side counts the number of subsets of size k and sums them up. □

Alternative proof of Proposition 1.5. The left hand side counts the number of all size-k subsets of $[n]$. The right hand side counts the same number by dividing subsets into two parts: $\binom{n-1}{k-1}$ subsets containing n and $\binom{n-1}{k}$ subsets not containing n. □

This kind of proofs is called *combinatorial proofs*, or *double counting*. To prove a combinatorial identity, we find a counting problem that can be able to answer in two ways, and then explain that the left hand side counts in one way while the right hand side counts in the other way.

Proposition 1.8.

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Proof. The right hand side is the number of ways of picking n unordered balls from $2n$ distinct balls. The left hand side is

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n}\binom{n}{0} = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k},$$

counting the same number by enumerating the number of balls that we pick from the first n balls. □

In fact, it is a special case of the following *Vandermonde's identity*, or *Vandermonde's convolution formula*.

It is named after Alexandre-Théophile Vandermonde.

Theorem 1.9 (Vandermonde's identity)**.**

$$\binom{n+m}{k} = \sum_{r=0}^{k}\binom{m}{r}\binom{n}{k-r}.$$

Proposition 1.10.

$$\sum_{m=0}^{n}\binom{m}{k}\binom{n-m}{r} = \binom{n+1}{k+r+1}.$$

Proof. The left hand side partitions all possibilities by enumerating the position of the $(k+1)$-th ball. □

Our last two examples involves Stirling numbers of the second kind.

Proposition 1.11.

$$\left\{{n \atop m}\right\} = \sum_{k=0}^{n-1}\binom{n-1}{k}\left\{{n-k-1 \atop k-1}\right\}.$$

Proof. Recall that the left hand side counts the number of ways to partition $[n]$ into m nonempty subsets. The right hand side counts the same number by enumerating all elements in the same subset as n. □

Proposition 1.12.

$$m^n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} (m)_k \,.$$

Proof. Recall the *twelvefold way*. The left hand side counts the number in Case 1, while the right hand side counts the same number by enumerating nonempty urns and then applying Case 9. □

1.4 *Catalan numbers*

We now introduce an example with a more complicated combinatorial proof.

Question 1.13. In a $n \times n$ grid, how many paths from the bottom left corner $(0, 0)$ to the top right corner (n, n), consisting entirely of edges pointing rightwards or upwards?

Clearly the answer is binomial coefficient $\binom{2n}{n}$. Now we add some restrictions.

This is a.k.a the *Dyck path*.

Question 1.14. In a $n \times n$ grid, how many paths from the left bottom corner to the right top corner, consisting entirely of edges pointing rightwards or upwards, which do not pass above the diagonal?

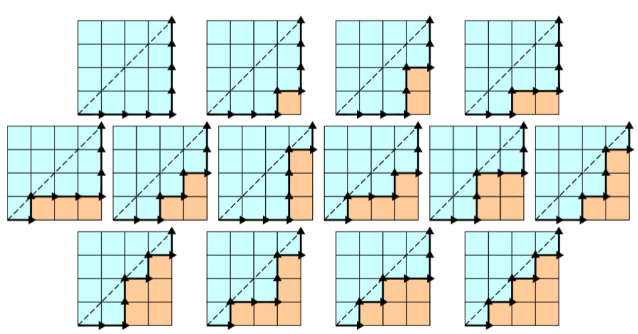


Figure 1.2: Diagram for the case $n = 4$, from wikipedia

The answers to this question are called *Catalan numbers*, where the expression of the n-th Catalan number is

$$C_n = \frac{1}{n+1} \binom{2n}{n} \,.$$

The sequence is named after Eugène Charles Catalan

The first Catalan numbers for $n = 0, 1, 2, 3, 4, 5, \ldots$ are

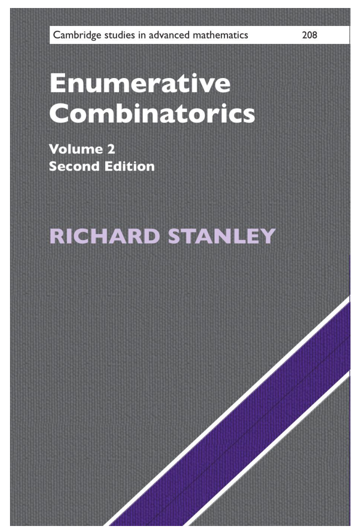
$$1,\ 1,\ 2,\ 5,\ 14,\ 42,\ \ldots$$

We now prove the formula of Catalan numbers.

Proof. We first claim that the number of invalid paths that pass above the diagonal is $\binom{2n}{n+1}$. Thus, the n-th Catalan number is given by

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}.$$

To prove our claim, consider an invalid path from $(0,0)$ to (n,n) that passes above the diagonal. There is a smallest k that after the k-th edge the path crosses the diagonal. By the minimality of k, the position after the k-th edge is $(\frac{k-1}{2}, \frac{k+1}{2})$. Now consider the effect of *reversing* the direction of each of the next $2n-k$ edges, namely, rightwards edges point upwards and upwards edges point rightwards. Originally, the next $2n-k$ edges go from $(\frac{k-1}{2}, \frac{k+1}{2})$ to (n,n). After reversing, they will terminate at $(n-1, n+1)$. Note that this construction is a *bijection*. For any path from $(0,0)$ to $(n-1, n+1)$, it must cross the diagonal of $n \times n$ grid. Reversing again the direction of remaining edges after the first edge that passes above the diagonal, we can obtain an invalid path from $(0,0)$ to (n,n). Thus, the number of invalid paths that cross the diagonal is exactly the number of paths from $(0,0)$ to $(n-1, n+1)$, which is $\binom{2n}{n+1}$. □



The book *Enumerative Combinatorics: Volume 2* by combinatorialist Richard P. Stanley contains a set of exercises which describe 66 different interpretations of the Catalan numbers.

There are many counting problems in combinatorics whose solution is given by the Catalan numbers. Here we give some examples.

Example 1.15 (Parenthesis sequences)**.** A valid parenthesis sequence is one in which every opening parenthesis "(" is matched with a corresponding closing parenthesis ")". These sequences are also known as *balanced parentheses sequences*. The number of valid parenthesis sequences of length $2n$ is C_n.

Each valid sequence corresponds to a path in a grid of $n \times n$ squares from the bottom left corner to the top right corner, where moving up represents adding an open parenthesis "(" and moving right represents adding a close parenthesis ")".

Example 1.16 (Push-pop sequences)**.** A push-pop sequence of a stack is a sequence of operations where you can push elements onto the stack and pop them off the stack. The number of ways to perform push and pop operations on a stack without violating the LIFO (Last-In-First-Out) property is equivalent to the number of valid parenthesis sequences.

Example 1.17 (Ordered trees)**.** An ordered tree, also known as a rooted tree, is a tree in which one vertex is designated as the root, and every other vertex has a unique parent vertex. The vertices of the tree are not labeled, but the children of each vertex (if exists) are ordered. Then the number of ordered trees of size $n+1$ is C_n, because of the bijection of the DFS sequences starting from the root and the push-pop sequences of a stack.

To see more appearance of the Catalan numbers, a useful tool is the following recurrence relation of the Catalan numbers.

Theorem 1.18. *The Catalan numbers satisfy the recurrence relation*

$$C_{n+1} = C_0 C_n + C_1 C_{n-1} + \cdots + C_n C_0 = \sum_{k=0}^{n} C_k C_{n-k}\,.$$

Proof. Consider a Dyck path of length $2(n+1)$. Let $k+1$ be the first nonzero x-coordinate where the path hits the diagonal, then $0 \le k < n$. The path breaks up into two pieces, the part to the left of $2(k+1)$ and the part to its right. The part to the right is a Dyck path of length $2(n-k)$, so it is counted by C_{n-k}. The left part is a path from $(0,0)$ to $(k+1,k+1)$ that does not pass above or hit the diagonal. So it must pass through $(1,0)$ and $(k+1,k)$, and between these two points, it never goes over the diagonal from $(1,0)$ to $(k+1,k)$, that is, a rightwards edge, then a Dyck path of length $2k$, and then an upwards edge. Hence, there are a total of $C_k C_{n-k}$ paths so that the first hit with the diagonal is at position $(k+1,k+1)$. Summing these terms up gives the recurrence relation. □

This recurrence can give more examples of the Catalan numbers.

Example 1.19 (Binary trees)**.** The number of different forms of binary trees with n vertices is C_n, since they satisfy the same recurrence relation.

It also describes the number of *full binary trees* with n internal vertices.

Example 1.20 (Polygon triangulations)**.** A *convex polygon triangulation* is to cut a convex polygon with $n+2$ sides into n triangles, by drawing $n-1$ non-crossing lines between vertices of the polygon. The number of convex polygon triangulation of an $(n+2)$-gon is C_n.

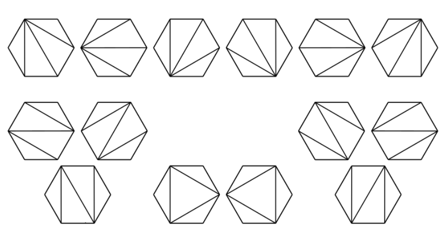


Figure 1.3: All triangulations for $n = 4$