2

Generating Functions

Recall that in Chapter 1, we introduce the following equation

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k ,$$

and use it to prove some properties of binomial coefficients, such as Propositions 1.3 and 1.4. Note that we transform binomial coefficients into a function, allowing us to manipulate and analyze sequences more effectively. This tool is called *generating functions*. Actually, in addition to substitution, we can perform more algebraic operations, such as addition, multiplication, and composition, which makes generating functions a powerful tool to solve counting problems, find closed-form expressions for sequences, derive recurrence relations, and analyze the behavior of combinatorial structures.

2.1 *Ordinary generating functions*

Definition 2.1 (Ordinary generating function)**.** Given a sequence $\{a_n\}_{n\geq 0}$, the *ordinary generating function* (OGF) defined by $\{a_n\}$ is

$$G(x) = \sum_{n\geq 0} a_n x^n .$$

Remark 2.2. You may argue that OGF is not well-defined since for some sequences, such a definition would lead to non-convergence. In fact, the generating function is not actually regarded as a function. It is a *formal* power series, and is not required to converge.

Example 2.3. Here are some introductory examples of generating functions:

1. Fix $m \in \mathbb{N}_+$ and let $a_n = \binom{m}{n}$ for all $n \geq 0$. Then the corresponding generating function is $G(x) = \sum_{n\geq 0} \binom{m}{n} x^n = (1+x)^m$.

2. Suppose $\{a_n\}$ is a constant sequence, e.g., $a_n = 1$ for all $n \geq 0$. Then the corresponding generating function is
$$G(x) = 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}\,.$$
3. Suppose $\{a_n\}$ is a geometric progression, e.g., $a_n = \alpha^n$ for some constant α. Then the corresponding generating function is
$$G(x) = 1 + \alpha x + \alpha^2 x^2 + \alpha^3 x^3 + \cdots = \frac{1}{1-\alpha x}\,.$$

Again, you may argue that the equality holds only if $|x| < 1$. Indeed, the closed form expression can often be interpreted as a function that can be evaluated at (sufficiently small) concrete values of x, and which has the formal series as its series expansion. However such interpretation is not required to be possible, and we can manipulate the closed form expression without worrying about convergence.

Given a sequence, it is easy to construct its generating function. Although it is not easy to find its closed-form expression, we do not usually need to do so. Conversely, if we are given a (closed-form) generating function, how can we know its corresponding sequences? The basic idea is to apply the well-known geometric series
$$\frac{1}{1-x} = \sum_{n\geq 0} x^n\,.$$
It is useful when we can express the generating function in the form of
$$G(x) = \frac{a_1}{1-b_1x} + \frac{a_2}{1-b_2x} + \cdots + \frac{a_k}{1-b_kx}\,.$$
The coefficient of the x^n-term is $[x^n]G(x) = a_1b_1^n + a_2b_2^n + \cdots + a_kb_k^n$. Generally, in principle, we can always use the Taylor series

Give a polynomial $G(x)$, denote by $[x^n]G(x)$ the coefficient of x^n term in $G(x)$.

$$G(x) = \sum_{n\geq 0} \frac{G^{(n)}(0)}{n!}x^n\,,$$
where $G^{(n)}(0)$ is the value of the n-th derivative of evaluated at $x = 0$. In particular, note that the n-th derivative of $(1+x)^r$ for some real r is
$$r(r-1)(r-2)\cdots(r-n+1)(1+x)^{n-r} = (r)_n(1+x)^{n-r}\,.$$
Using the (generalized) definition of binomial coefficients, we have the following theorem.

Actually, the same generalization also applies to complex exponents.

Theorem 2.4 (Newton's generalized binomial theorem). *If x is any real number with $|x| < 1$ and r is any complex number, we have*
$$(1+x)^r = \sum_{n=0}^{\infty}\binom{r}{n}x^n\,,$$
where $\binom{r}{n} = \frac{(r)_n}{n!}$.

Example 2.5. For example, we have
$$\frac{1}{1+x} = (1+x)^{-1} = \sum_{n\geq 0}\frac{(-1)^n n!}{n!}x^n = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots\,.$$

2.2 *Operations on generating functions*

Translating sequences into polynomials is advantageous because polynomial multiplication, in a sense, encodes both the multiplication principle and the addition principle. Let $F(x) = \sum_{n\geq 0} f_n x^n$ and $G(x) = \sum_{n\geq 0} g_n x^n$. Then

$$[x^n]\Big(F(x)G(x)\Big) = \sum_{k=0}^{n} f_k g_{n-k}\,.$$

Formally, the corresponding operations on coefficients is called *convolution*. It has a clear combinatorial meaning. Suppose we have two disjoint sets $\mathcal{F}$ and $\mathcal{G}$, and there are f_n and g_n ways to pick n elements from $\mathcal{F}$ and $\mathcal{G}$, respectively. Then how many ways could we pick n elements from $\mathcal{F} \cup \mathcal{G}$? The answer is the *convolution* of $\{f_n\}$ and $\{g_n\}$, since we can first enumerate the number of elements from $\mathcal{F}$, then count the number of ways to pick k elements from $\mathcal{F}$ and $n-k$ elements from $\mathcal{G}$, and finally sum them up.

For example, suppose there are 5 (identical) blue balls, 3 (identical) green balls, and 2 (identical) red balls. How many ways can we pick 6 balls among them? Of course you can count by enumerating all possibilities. Now we apply generating functions. Let $\{b_n\}$, $\{g_n\}$, $\{r_n\}$ be the sequences of numbers of ways to pick blue, green, and red balls, respectively. Then their corresponding generating functions are

$$\begin{aligned} B(x) &= 1 + x + x^2 + x^3 + x^4 + x^5\,, \\ G(x) &= 1 + x + x^2 + x^3\,, \\ R(x) &= 1 + x + x^2\,. \end{aligned}$$

Then we let

$$\begin{aligned} F(x) &= B(x) \cdot G(x) \cdot R(x) \\ &= (1 + x + x^2 + x^3 + x^4 + x^5) \cdot (1 + x + x^2 + x^3) \cdot (1 + x + x^2) \\ &= 1 + 3x + 6x^2 + 9x^3 + 11x^4 + 12x^5 + 11x^6 + 9x^7 + 6x^8 + 3x^9 + x^{10} \end{aligned}$$

and $[x^n]F(x)$ gives the numbere of ways to pick n balls among all 10 balls.

Now we can give an alternate proof of the claim in Case 7 of the *twelvefold way*: the multiset number $\left(\!\binom{m}{n}\!\right)$ is equal to $\binom{n+m-1}{n}$.

Proposition 2.6.

$$\left(\!\!\binom{m}{n}\!\!\right) = \binom{n+m-1}{n} = \binom{n+m-1}{m-1}.$$

Proof. Recall that $\left(\!\binom{m}{n}\!\right)$ is the number of multisets (allowing multiple copies of each elements) of size n, with elements taken from $[m]$. Let $S_i(x)$ be the generating function of the number of ways to select n copies of element i. Clearly,

$$S_i(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

Thus the generating function of $\left(\!\binom{m}{n}\!\right)$ is

$$\begin{aligned} S(x) &\triangleq \sum_{n\geq 0} \left(\!\binom{m}{n}\!\right) x^n \\ &= S_1(x)\cdot S_2(x)\cdot\cdots\cdot S_m(x) \\ &= \frac{1}{1-x}\cdot\frac{1}{1-x}\cdot\cdots\cdot\frac{1}{1-x} \\ &= (1-x)^{-m}. \end{aligned}$$

Applying Theorem 2.4, we conclude that

$$\begin{aligned} \left(\!\binom{m}{n}\!\right) &= [x^n](1-x)^{-m} = \frac{(-m)_n}{n!}(-1)^n \\ &= \frac{(-m)(-m-1)\cdots(-m-n+1)}{n!}(-1)^n \\ &= \frac{m(m+1)\cdots(m+n-1)}{n!} = \binom{n+m-1}{n}. \end{aligned}$$

□

Our final example of multiplication is the following theorem proved by Leonhard Euler in 1748.

Theorem 2.7. *For each positive n, the number of partitions of n into odd parts equals the number of partitions of n into distinct parts.*

Could you give a combinatorial proof?

Proof. Let o_n be the number of ways to partition n into odd parts, d_n be the number of ways to partition n into distinct parts, and let $O(x), D(x)$ be their generating functions respectively. For odd parts, we have

$$\begin{aligned} O(x) &= (1+x+x^2+\cdots)\big(1+x^3+(x^3)^2+\cdots\big)\big(1+x^5+(x^5)^2+\cdots\big)\cdots \\ &= \frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdots \\ &= \prod_{k \bmod 2=1}\frac{1}{1-x^k}. \end{aligned}$$

For distinct parts, we have

$$\begin{aligned} D(x) &= (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\cdots \\ &= \frac{1-x^2}{1-x}\cdot\frac{1-x^4}{1-x^2}\cdot\frac{1-x^6}{1-x^3}\cdot\frac{1-x^8}{1-x^4}\cdot\frac{1-x^{10}}{1-x^5}\cdots \\ &= \prod_{k \bmod 2=1}\frac{1}{1-x^k}. \end{aligned}$$

So we conclude that $O(x) = D(x)$, and thus $o_n = d_n$ for all $n \geq 0$. $\square$

In addition to multiplication, there are more interesting operations on generating functions, such as differentiation.

Proposition 2.8.

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = \sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Could you give a combinatorial proof?

Proof. Let $G(x) = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. Then $G'(x) = n(1+x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k} x^{k-1}$. Substitute $x = 1$ into it. $\square$

We will not give more examples of differentiation, because we are not going to involve differential equations in this course.

Let $F(x) = \sum_{n\geq 0} f_n x^n$ and $G(x) = \sum_{n\geq 0} g_n x^n$. Here is a table summarizing the common operations on the generating function and their corresponding effects on the sequence.

sequence	generating functions
shift	multiple (by x^k): $x^k G(x) = \sum_{n\geq k} g_{n-k} x^n$
addition	addition: $F(x) + G(x) = \sum_{n\geq 0} (f_n + g_n) x^n$
convolution	multiplication: $F(x)G(x) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} f_k g_{n-k} \right) x^n$
multiple (by n)	differentiation: $G'(x) = \sum_{n\geq 0} (n+1) g_{n+1} x^n$

2.3 *Solving recurrence*

One of the most important applications of generating functions is to solve recurrence and find closed form. Now we introduce some examples.

The well-known *Fibonacci sequence* $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ has the following recurrence relation:

$$f_0 = 0, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2.$$

Let $F(x)$ be its generating function. Then we have

$$\begin{aligned}
F(x) &= f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + f_5 x^5 + \cdots, \\
xF(x) &= \phantom{f_0 + {}} f_0 x + f_1 x^2 + f_2 x^3 + f_3 x^4 + f_4 x^5 + \cdots, \\
x^2 F(x) &= \phantom{f_0 + f_1 x + {}} f_0 x^2 + f_1 x^3 + f_2 x^4 + f_3 x^5 + \cdots.
\end{aligned}$$

By the recurrence, it follows that

$$F(x) = xF(x) + x^2F(x) + x\,,$$

and thus $F(x) = \frac{x}{1-x-x^2}$. To find the closed form of f_n, we hope $F(x)$ have the form

$$F(x) = \frac{a_1}{1-b_1x} + \frac{a_2}{1-b_2x}\,.$$

Solving the equation

$$(1-b_1x)(1-b_2x) = 1-x-x^2$$

we obtain $b_1 = \frac{1}{2}(1+\sqrt{5})$ and $b_2 = \frac{1}{2}(1-\sqrt{5})$. Solving the equation

$$a_1(1-b_2x) + a_2(1-b_1x) = x$$

we obtain $a_1 = \frac{1}{\sqrt{5}}$ and $a_2 = -\frac{1}{\sqrt{5}}$. Hence, the closed form of f_n is

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Next, we revisit the Catalan numbers. Recall that, the recurrence relation for Catalan numbers is (cf. Theorem 1.18)

$$C_0 = 1\,, \quad C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k} \text{ for all } n \geq 1\,.$$

Let $G(x) = C_0 + C_1x + C_2x^2 + \cdots$ be the generating function of Catalan numbers. The recurrence relation reveals that we should consider multiplication. Since

$$\begin{aligned}
G(x) &= C_0 + \sum_{n\geq 1}\sum_{k=0}^{n-1} C_k C_{n-1-k} x^n\,,\\
G(x)^2 &= \Big(\sum_{n\geq 0} C_n x^n\Big)^2 = \sum_{n\geq 0}\sum_{k=0}^{n} C_k C_{n-k} x^n\,,\\
xG(x)^2 &= \sum_{n\geq 1}\sum_{k=0}^{n-1} C_k C_{n-1-k} x^n\,,
\end{aligned}$$

we have $G(x) = xG(x)^2 + 1$. Thus, solving $xG(x)^2 - G(x) + 1 = 0$, we obtain

$$G(x) = \frac{1 \pm \sqrt{1-4x}}{2x}\,.$$

Only one of these solutions can be the generating function. Noting that

$$\lim_{x\to 0} G(x) = G(0) = C_0 = 1\,,$$

it is easy to check that $G(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the correct one. Now we expand $1-\sqrt{1-4x}$ by Netwon' generalized binomial formula:

$$\begin{aligned} 1-(1-4x)^{1/2} &= 1-\sum_{n\geq 0}\binom{1/2}{n}(-4x)^n \\ &= 1-1-\sum_{n\geq 1}\binom{1/2}{n}(-4x)^n \\ &= 4x\sum_{n\geq 0}\binom{1/2}{n+1}(-4x)^n . \end{aligned}$$

Thus, it follows that

$$G(x) = 2\sum_{n\geq 0}\binom{1/2}{n+1}(-4x)^n ,$$

and then

$$\begin{aligned} C_n &= 2\binom{1/2}{n+1}(-4)^n = \frac{2\cdot(-4)^n}{(n+1)!}\left(\frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}\cdot\dots\cdot\frac{-(2n-1)}{2}\right) \\ &= \frac{2\cdot(-4)^n}{(n+1)!}\cdot\frac{(-1)^n(2n-1)!!}{2^{n+1}} = \frac{2^n}{(n+1)!}\cdot\frac{(2n)!}{n!2^n} = \frac{1}{n+1}\binom{2n}{n} . \end{aligned}$$

Our final example is the Stirling numbers of the second kind, with the recurrence relation

$$\left\{ {0 \atop 0} \right\} = 1 , \quad \left\{ {n \atop k} \right\} = k\left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\} \text{ for } (n,k) \neq (0,0) .$$

Since there are 2 indices, it is easy to find three natural candidates for generating functions:

$$\begin{aligned} A(x,y) &= \sum_{n\geq 0}\sum_{k\geq 0}\left\{ {n \atop k} \right\} x^n y^k , \\ B_k(x) &= \sum_{n\geq 0}\left\{ {n \atop k} \right\} x^n , \\ C_n(y) &= \sum_{k\geq 0}\left\{ {n \atop k} \right\} y^k . \end{aligned}$$

Let's develop some intuitions about which choice is likely to succeed. First, $A(x,y)$ is ruled out since we do not know how to deal with multivariate generating functions. Then, if we choose $C_n(y)$, the term of $k\{ {n-1 \atop k} \}$ in recurrence relation is related to differentiation, which becomes more complicated. So we try to find the function $B_k(x)$. Note that

$$\begin{aligned} B_k(x) &= \left\{ {0 \atop k} \right\} + \left\{ {1 \atop k} \right\} x + \left\{ {2 \atop k} \right\} x^2 + \left\{ {3 \atop k} \right\} x^3 + \cdots , \\ kxB_k(x) &= \qquad k\left\{ {0 \atop k} \right\} x + k\left\{ {1 \atop k} \right\} x^2 + k\left\{ {2 \atop k} \right\} x^3 + \cdots , \\ xB_{k-1}(x) &= \qquad \left\{ {0 \atop k-1} \right\} x + \left\{ {1 \atop k-1} \right\} x^2 + \left\{ {2 \atop k-1} \right\} x^3 + \cdots . \end{aligned}$$

So it is easy to find that

$$B_k(x) = kxB_k(x) + xB_{k-1}(x)$$

for all $k \geq 1$, and $B_0(x) = 1$. This leads to

$$\begin{aligned} B_k(x) &= \frac{x}{1-kx} \cdot B_{k-1}(x) = \frac{x}{1-kx} \cdot \frac{x}{1-(k-1)x} \cdot B_{k-2}(x) \\ &= \cdots = \frac{x^k}{(1-x)(1-2x)(1-3x)\cdots(1-kx)} \,. \end{aligned}$$

Our goal is to find an explicit formula of $[x^n]B_k(x)$. A natural idea is to rewrite $B_k(x)$ in the form of

$$B_k(x) = \prod_{i=1}^{k} \frac{x}{1-i\cdot x} = \sum_{i=1}^{k} \frac{r_i \cdot x}{1-i\cdot x} \,.$$

To find r_i's, we fix some $j \in [k]$ and multiply both sides by $(1 - j \cdot x)$. It gives that

$$x \cdot \prod_{i\neq j} \frac{x}{1-i\cdot x} = r_j \cdot x + \sum_{i\neq j} \frac{1-j\cdot x}{1-i\cdot x} \cdot r_i \cdot x \,.$$

Then let $x = 1/j$ and we can get that

$$r_j = \prod_{i\neq j} \frac{1/j}{1-i/j} = \prod_{i\neq j} \frac{1}{j-i} = \frac{(-1)^{k-j}}{(j-1)!(k-j)!} = (-1)^{k-j} \frac{j}{k!} \binom{k}{j} \,.$$

Thus, it follows that

$$\begin{aligned} \left\{ {n \atop k} \right\} = [x^n]B_k(x) &= \sum_{i=1}^{k} [x^n] \frac{r_i \cdot x}{1-i\cdot x} \\ &= \sum_{i=1}^{k} [x^{n-1}] \frac{r_i}{1-i\cdot x} \\ &= \sum_{i=1}^{k} r_i \cdot i^{n-1} \\ &= \frac{1}{k!} \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} i^n \,. \end{aligned}$$

2.4 *Exponential generating functions*

Usually, ordinary generating functions is useful for counting the number of subsets. However, it may not work for counting permutations or ordered / labeled elements. For example, what is the (ordinary) generating function for the number of permutations on $[n]$? Clearly we have

$$F(x) = 1 + x + 2x^2 + 6x^3 + \cdots = \sum_{n\geq 0} n! x^n \,.$$

Now we try to find its closed-form expression. Since $[x^n]F(x) = n[x^{n-1}]F(x)$, it suggests the differentiation operation:

$$\begin{aligned} F(x) &= 1 + x + 2x^2 + 6x^3 + 24x^4 + \cdots, \\ xF(x) &= x + x^2 + 2x^3 + 6x^4 + 24x^5 + \cdots, \\ x(xF(x))' &= \quad x + 2x^2 + 6x^3 + 24x^4 + \cdots. \end{aligned}$$

Thus we have

$$F(x) = 1 + x(xF(x))' = 1 + xF(x) + x^2F'(x).$$

Unfortunately, such type of differential equations does not have closed-form solution in general.

We now introduce *exponential generating functions*, which is used to deal with counting permutations or labeled elements.

Definition 2.9 (Exponential generating function)**.** Given a sequence $\{a_n\}_{n\geq 0}$, the *exponential generating function* (EGF) defined by $\{a_n\}$ is

$$\hat{G}(x) = \sum_{n\geq 0} \frac{a_n}{n!} x^n.$$

Example 2.10 (Permutation)**.** It is easy to see that the EGF defined by $n!$ is

$$\hat{F}(x) = \sum_{n\geq 0} x^n = \frac{1}{1-x}.$$

Example 2.11 (Circular permutation)**.** The number of ways to arrange n ($n \geq 1$) distinct objects along a fixed circle is $P_n = (n-1)!$. It EGF is given by

$$\hat{P}(x) = \sum_{n\geq 1} \frac{(n-1)!}{n!} x^n = \sum_{n\geq 1} \frac{x^n}{n}.$$

Note that $\hat{P}(0) = 0$ and $\hat{P}'(x) = \sum_{n\geq 1} x^{n-1}$. So we find that

Note that $\hat{G}(x) = \exp \hat{P}(x)$. It is NOT a coincidence!

$$\hat{P}(x) = \int \left(\sum_{n\geq 0} x^n\right) \mathrm{d}x = \int \frac{1}{1-x} \mathrm{d}x = \ln \frac{1}{1-x}.$$

To see why exponential generating functions counts the number of ordered or labeled elements, we multiply two generating functions again and see what happens to their coefficients. Let

$$\hat{F}(x) = \sum_{n\geq 0} \frac{f_n}{n!} x^n, \quad \text{and} \quad \hat{G}(x) = \sum_{n\geq 0} \frac{g_n}{n!} x^n.$$

Then, it is easy to see that

$$[x^n]\Big(\hat{F}(x)\hat{G}(x)\Big) = \sum_{k=0}^{n} \frac{f_k}{k!} \cdot \frac{g_{n-k}}{(n-k)!},$$

and thus

$$h_n = n! \cdot [x^n]\Big(\hat{F}(x)\hat{G}(x)\Big) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} f_k\, g_{n-k} = \sum_{k=0}^{n} \binom{n}{k} f_k\, g_{n-k}\,.$$

Here $\{h_n\}$ is the corresponding sequence of the exponential generating function $\big(\hat{F}(x)\hat{G}(x)\big)$. Again, suppose we have two disjoint sets $\mathcal{F}$ and $\mathcal{G}$, and we would like to count the number of ways to pick n elements from $\mathcal{F} \cup \mathcal{G}$. But now we enumerate not only the number of elements chosen from $\mathcal{F}$, but also the *positions* or *labels* of them.

Moreover, unlike the operations on ordinary generating functions, now $x\hat{G}(x)$ corresponds the product ng_{n-1}, and the differentiation $\hat{G}'(x)$ corresponds the shift of $\{g_n\}_{n\geq 0}$.

We now consider an application of exponential generating functions. Suppose there is a $1 \times n$ board, and we would like to color each square with color blue, green, and red. It is required that the number of red squares is even and there is at least one blue square. Our goal is to determine the number f_n of ways to color the board.

Let $\{b_n\}$, $\{g_n\}$, $\{r_n\}$ be the numbers of ways to color n squares with single color blue, green, and red, respectively. Then their exponential generating functions are

$$\begin{aligned}
\hat{B}(x) &= \phantom{1+{}} \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = e^x - 1\,,\\
\hat{G}(x) &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = e^x\,,\\
\hat{R}(x) &= 1 \phantom{{}+\frac{x}{1!}} + \frac{x^2}{2!} \phantom{{}+\frac{x^3}{3!}} + \frac{x^4}{4!} \phantom{{}+\frac{x^5}{5!}} + \cdots = \frac{e^x + e^{-x}}{2}\,.
\end{aligned}$$

Thus the exponential generating function of $\{f_n\}$ is

$$\begin{aligned}
\hat{F}(x) &= \hat{B}(x) \cdot \hat{G}(x) \cdot \hat{R}(x)\\
&= (e^x - 1) \cdot e^x \cdot \frac{e^x + e^{-x}}{2}\\
&= \frac{e^{3x} - e^{2x} + e^x - 1}{2}\\
&= -\frac{1}{2} + \sum_{n\geq 0} \frac{3^n - 2^n + 1}{2} \cdot \frac{x^n}{n!}\,,
\end{aligned}$$

which gives that

$$f_n = n! \cdot [x^n]\hat{F}(x) = \begin{cases} 0 & n = 0\,,\\ \frac{3^n - 2^n + 1}{2} & n \geq 1\,. \end{cases}$$

Similarly, recall that, $\left\{ {n \atop k} \right\}$ counts the number of ways to partition $[n]$ into k identical nonempty subsets. So $k!\left\{ {n \atop k} \right\}$ has the exponential generating function

$$\sum_{n\geq 0} \frac{k!}{n!} \left\{ {n \atop k} \right\} x^n = \left(e^x - 1\right)^k.$$