# 4
# Partially Ordered Sets

We introduce a simple structure: a set associated with a partial order.

## 4.1 Definitions of posets and lattices

**Definition 4.1** (Partially ordered set, poset)**.** A partially ordered set (or *poset*) is a pair $\mathcal{P} = (P, \preccurlyeq)$, where $P$ is a set and $\preccurlyeq$ is a *partial order*, i.e., a binary relation satisfying

1. (*reflexivity*) $\forall\, x \in P$, $x \preccurlyeq P$;
2. (*anti-symmetry*) $\forall\, x, y \in P$, if $x \preccurlyeq y$ and $y \preccurlyeq x$, then $x = y$;
3. (*transitivity*) $\forall\, x, y, z \in P$, if $x \preccurlyeq y$ and $y \preccurlyeq$, then $x \preccurlyeq z$.

In fact, a partial order is a generalization of $\leq$, but we do not require any pair of elements is comparable. In particular, if $x \preccurlyeq y$ but $x \neq y$, we denote $x \prec y$.

**Example 4.2.** $(P, \leq)$ is a poset, where $P \subseteq \mathbb{Z}$ and $\leq$ is the usual integer order.

**Example 4.3.** $(2^{[n]}, \subseteq)$ is a poset, where $\subseteq$ is the usual set inclusion.

**Example 4.4.** $([n], |)$ is a poset, where $|$ is the divisibility relation.

Denote by $[x, y]$ the set $\{z \mid x \preccurlyeq z \preccurlyeq y\}$, and denote by $x \lessdot y$ if $[x, y] = \{x, y\}$. Then a poset can be expressed as a *Hasse diagram*, where we draw a line between $x$ and $y$ (with $y$ above $x$) if $x \lessdot y$.

Sometimes we call it "$y$ covers $x$".

**Exercise 4.5.** *Draw Hasse diagrams of the subset poset and the divisibility poset.*

We say $x \in P$ is *maximal* if there is no $y \in P$ such that $x \preccurlyeq y$; and $x$ is *maximum* if for all $y \in P$, $y \preccurlyeq x$. We can also define maximal elements and the maximum element in a subset of $P$. Similarly we can define *minimal* and the *minimum* element in $P$ or a subset of $P$.

Why is the maximum element unique?

**Example 4.6.** A poset may have *maximal* elements but no *maximum* elements, e.g., $([6], |)$.

**Example 4.7.** A poset may have no maximal elements, e.g., $(\mathbb{N}, \leq)$.

Given a poset $\mathcal{P} = (P, \preccurlyeq)$ and a subset $S \subseteq P$, $x \in P$ is an *upper bound* of $S$ if for any $y \in S$, $y \preccurlyeq x$. The *minimum* element (if exists) of upper bounds of $S$ is called the *least upper bound* of $S$. Similarly, we can define *lower bounds* of a subset, and the *greatest lower bound*.

In a poset, the least upper bound or the greatest lower bound of some subset $S$ may not exist. If exist, the poset is called a *lattice*.

**Definition 4.8** (Lattice)**.** A *lattice* $\mathcal{L} = (P, \preccurlyeq)$ is a poset where for any *nonempty finite* subset $S \subseteq P$, its least upper bound and greatest lower bound exist.
In fact, a lattice can also be defined as a poset where any pair $x, y \in P$ has the least upper bound and the greatest lower bound. The least upper bound of $x$ and $y$, a.k.a. the *join* of $x$ and $y$, is denoted by $x \vee y$. The greatest lower bound of $x$ and $y$, a.k.a. the *meet* of $x$ and $y$, is denoted by $x \wedge y$.

**Example 4.9.** $(\mathbb{N}, |)$ is a lattice, where $x \vee y = \mathrm{lcm}(x, y)$ and $x \wedge y = \gcd(x, y)$.

**Example 4.10.** $(2^{[n]}, \subseteq)$ is a lattice, where $S \vee T = S \cup T$ and $S \wedge T = S \cap T$.

## 4.2 *Chains and antichains*

**Definition 4.11** (Chain and antichain)**.** A *chain* in a poset $(P, \preccurlyeq)$ is a subset $C \subseteq P$ where any two elements in $C$ are comparable, namely,

$$\forall\, x, y \in C\,, \qquad x \preccurlyeq y \;\vee\; y \preccurlyeq x\,.$$

A *antichain* is a subset $T \subseteq P$ where any two distinct elements are not comparable, namely,

$$\forall\, x \neq y \in T\,, \qquad x \not\preccurlyeq y \;\wedge\; y \not\preccurlyeq x\,.$$

The *height* of a poset is the maximum size of a chain in the poset. The *width* of a poset is the maximum size of an antichain in the poset.

**Theorem 4.12** (Mirsky's theorem)**.** *Let $\mathcal{P}$ be a poset with a finite height. The minimum number of antichains that can cover $\mathcal{P}$ is the height of $\mathcal{P}$.*

**Example 4.13.** Suppose $S = (s_1, s_2, \ldots, s_n)$ is a sequence. The minimum number of decreasing subsequences to partition $S$ is exactly the length of the longest increasing (nondecreasing) subsequence.

This example yields the following corollary.

**Corollary 4.14** (Erdős-Szekeres theorem)**.** *Suppose $r, s \in \mathbb{N}, n \geq rs + 1$, and $S = (s_1, s_2, \ldots, s_n)$ is a sequence of $n$ distinct real numbers. Then $S$ contains an increasing (decreasing) subsequence of length $r + 1$ or a decreasing (increasing) subsequence of length $s + 1$.*

*Proof.* Let $\preccurlyeq$ be the relation such that $s_i \preccurlyeq s_j$ if $i \leq j$ and $s_i \leq s_j$. Then each increasing subsequence is a chain and each decreasing subsequence is an antichain. Suppose $S$ does not contain an increasing subsequence of length $r + 1$. Then by Mirsky's theorem, there are at most $r$ decreasing subsequences that can cover $S$. So the longest decreasing subsequence has length at least $s + 1$. $\square$

Now we prove Mirsky's theorem.

*Proof of Theorem 4.12.* Let $\mathcal{P}$ be a poset of height $h$, and $C$ be its maximum chain. Clearly any two elements in $C$ cannot be in the same antichain. So if we would like to cover $\mathcal{P}$ with antichains, there are at least $h$ of them.

Now we show that $h$ antichains are suffice. For each element $x \in \mathcal{P}$, let $f(x)$ be the maximum size of chains in which $x$ is the minimum element. A key observation is that if $f(x) = f(y)$, then $x$ and $y$ must be not comparable. Thus the set $f^{-1}(k)$ is an antichain, and $f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(h)$ are $h$ antichains that cover $\mathcal{P}$. $\square$

Conversely, we have a dual theorem of Mirsky's theorem.

**Theorem 4.15** (Dilworth's theorem)**.** *Let $\mathcal{P}$ be a poset with a finite width. The minimum number of chains that can cover $\mathcal{P}$ is the width of $\mathcal{P}$.*

We only prove Dilworth's theorem for finite size posets here. The generalized version requires De Bruijn-Erdős theorem, which states that an infinite graph can be colored with $c$ colors if the same is true for all its finite subgraphs.

*Proof.* Let $\mathcal{P} = (P, \preccurlyeq)$ be a poset of width $w$, and $T$ be its maximum antichain. Clearly any two elements in $T$ cannot be in the same chain. So if we would like to cover $\mathcal{P}$ with chains, there are at least $w$ of them.

Next we show that $w$ chains are suffice by the induction on $|P|$. If $|P| = 1$, it is obviously true. Now assume $|P| \geq 2$. If any two elements in $P$ are not comparable, then $\mathcal{P}$ has width $|P|$ and this case is trivial. Let $C$ be the maximum chain in $\mathcal{P}$, and $c^+, c^-$ be its maximum and minimum elements respectively. If $\mathcal{P} \setminus \{c^+, c^-\}$ has width $w - 1$, then by induction hypothesis, $\mathcal{P} \setminus \{c^+, c^-\}$ can be covered by $w - 1$

chains. Thus $\mathcal{P}$ can be covered by the same $w-1$ chains together with $\{c^+, c^-\}$. So we assume that the width of $\mathcal{P} \setminus \{c^+, c^-\}$ is also $w$.

Let $T = \{t_1, t_2, \dots, t_w\}$ be a maximum antichain of $\mathcal{P} \setminus \{c^+, c^-\}$. It is also a maximum antichain of $\mathcal{P}$. Partition $P \setminus T$ into 2 parts:

$$S^+ \triangleq \{s \in P \setminus T \mid \exists\, t \in T, t \preccurlyeq s\},$$
$$S^- \triangleq \{s \in P \setminus T \mid \exists\, t \in T, s \preccurlyeq t\}.$$

By the choice of $T$, $S^+ \uplus S^- = P \setminus T$. By the choice of $c^+$ and $c^-$, we have $c^+ \in S^+$ and $c^- \in S^-$. So both $S^+$ and $S^-$ are nonempty. Now applying induction hypothesis on $S^+ \cup T$, there are $w$ chains $C_1^+, C_2^+, \dots, C_w^+$ that cover $S^+ \cup T$ and $C_i^+$ contains $t_i$ as its minimum. Similarly, there are $C_1^-, C_2^-, \dots, C_w^-$ that cover $S^- \cup T$ and $C_i^-$ contains $t_i$ as its maximum. Hence,

The notation $\uplus$ means disjoint union, i.e., $S^+ \cup S^- = P \setminus T$ and $S^+ \cap S^- = \emptyset$.

$$C_1^+ \cup C_1^-,\ C_2^+ \cup C_2^-,\ \dots,\ C_w^+ \cup C_w^-$$

are $w$ chains that cover $P$. $\square$

In particular, by Dilworth's theorem, we can show that the width of poset $(2^{[n]}, \subseteq)$ is $\binom{n}{\lfloor n/2 \rfloor}$.

**Theorem 4.16** (Sperner's theorem)**.** *The maximum antichain in* $\mathcal{P} = (2^{[n]}, \subseteq)$ *has size* $\binom{n}{\lfloor n/2 \rfloor}$.

*Proof.* Let $m = \lfloor n/2 \rfloor$. Then $\binom{[n]}{m}$ is an antichain. Thus it suffices to show that $\binom{n}{m}$ chains can cover $\mathcal{P}$. Actually, it is sufficient to show that $\binom{n}{k}$ chains can cover $\binom{[n]}{k} \cup \binom{[n]}{k+1}$ if $k \geq m$, and $\binom{n}{k}$ chains can cover $\binom{[n]}{k} \cup \binom{[n]}{k-1}$ if $k \leq m$.

We only show the first case. The argument for the second case is similar. Applying Theorem 4.15 again, we need to show that the maximum antichain in $\binom{[n]}{k} \cup \binom{[n]}{k+1}$ has size $\binom{n}{k}$. Let $\mathcal{T}$ be any maximum antichain, $\mathcal{R} = \mathcal{T} \cap \binom{[n]}{k+1}$, and $\mathcal{S} = \binom{[n]}{k} \setminus \mathcal{T}$. Since $\mathcal{T}$ is an antichain, for any $R \in \mathcal{R}$, all size-$k$ subsets of $R$ must be in $\mathcal{S}$. Each $R \in \mathcal{R}$ has $(k+1)$ size-$k$ subsets, and each $S \in \mathcal{S}$ is a subset of at most $(n-k)$ sets in $\mathcal{R}$. Note that $k+1 \geq n-k$. Therefore, we have $|\mathcal{R}| \leq |\mathcal{S}|$, and thus $|\mathcal{T}| = \binom{n}{k} - |\mathcal{S}| + |\mathcal{R}| \leq \binom{n}{k}$. $\square$

In fact, Dilworth's theorem is equivalent to Hall's marriage theorem, and is also equivalent to König theorem. We will explain details in Chapter 6.

## 4.3 *Incidence algebra and Möbius inversion*

We would like to generalize Möbius inversion to posets. We first introduce *incidence algebra* on posets. Here posets may be infinite, but are required to be *locally finite*.

A poset $\mathcal{P}$ is locally finite if for any $x, y \in \mathcal{P}$, $[x, y]$ is finite.

**Definition 4.17** (Incidence algebra)**.** Let $\mathcal{P} = (P, \preccurlyeq)$ be a poset. Its *incidence algebra* is defined by

$$\mathcal{A}(\mathcal{P}) \triangleq \{\alpha : P \times P \to \mathbb{R} \mid \alpha(x, y) = 0 \text{ whenever } x \not\preccurlyeq y\} ,$$

namely, the set of functions on $P \times P$ whose nonzero values are all inside the $\preccurlyeq$ relation.

In particular, if $\mathcal{P}$ is finite, a function on $P \times P$ can be expressed as a $|P| \times |P|$ matrix, and thus $\mathcal{A}(\mathcal{P})$ is a set of matrices. Now we define operations on the incidence algebra, which are natural generalizations of operations on matrices.

**Definition 4.18.** Suppose $\alpha, \beta \in \mathcal{A}(\mathcal{P})$ are two functions. Then

- $(\alpha + \beta)(x, y) = \alpha(x, y) + \beta(x, y)$;
- $\forall\, c \in \mathbb{R}$, $(c\alpha)(x, y) = c \cdot \alpha(x, y)$;
- $(\alpha\beta)(x, y) = \sum_{z \in [x,y]} \alpha(x, z) \cdot \beta(z, y)$;
- $\forall\, f : P \to \mathbb{R}$, $(\alpha f)(x) = \sum_{x \preccurlyeq y} \alpha(x, y) \cdot f(y)$.

It is easy to verify that if $\alpha, \beta \in \mathcal{A}(\mathcal{P})$ and $c \in \mathbb{R}$, then $\alpha + \beta, c\alpha, \alpha\beta$ are all in $\mathcal{A}(\mathcal{P})$. Similar to identity matrices, we can define the *multiplicative identity* $\delta$ in $\mathcal{A}(\mathcal{P})$:

$$\delta(x, y) \triangleq [x = y] = \begin{cases} 1 & \text{if } x = y\,, \\ 0 & \text{if } x \neq y\,. \end{cases}$$

Given $\alpha \in \mathcal{A}(\mathcal{P})$, $\beta$ is a *left inverse* if $\beta\alpha = \delta$, and $\beta$ is a *right inverse* if $\alpha\beta = \delta$. A key fact is that if the left inverse exists then the right inverse exists, and they are the same function. Thus, we say $\beta$ is the *inverse* of $\alpha$, denoted by $\beta = \alpha^{-1}$, if $\alpha\beta = \delta$, or $\beta\alpha = \delta$.

**Theorem 4.19.** *Suppose $\alpha, \beta \in \mathcal{A}(\mathcal{P})$. If $\alpha\beta = \delta$, then $\beta\alpha = \delta$.*

*Proof.* For any $x \in P$, $(\beta\alpha)(x, x) = \beta(x, x)\, \alpha(x, x) = (\alpha\beta)(x, x) = 1$. So it suffices to show that $(\beta\alpha)(x, y) = 0$ if $x \prec y$.

Suppose $[x, y] = \{z_1, z_2, \ldots, z_n\}$ where $z_1 = x$ and $z_n = y$. Define two $n \times n$ matrices $A, B$ by $A_{i,j} = \alpha(z_i, z_j)$ and $B_{i,j} = \beta(z_i, z_j)$. Note that (by transitivity) $[z_i, z_j] \subseteq [x, y]$ for all $i, j$. Thus we have

$$(AB)_{i,j} = \sum_{k=1}^{n} A_{i,k} B_{k,j} = \sum_{z_k \in [z_i, z_j]} \alpha(z_i, z_k)\, \beta(z_k, z_j) = \delta(z_i, z_j) ,$$

that is, $AB = I$. So $BA = I$, which is equivalent to $(\beta\alpha)(z_i, z_j) = I_{i,j}$. In particular, $(\beta\alpha)(x, y) = 0$. $\square$

Moreover, it is easy to see that the inverse is unique. Suppose $\alpha\beta = \gamma\beta = \delta$. Then by the *associative law*, we have $\alpha = \alpha\beta\gamma = \gamma$.

Let $\mathcal{P}$ be a poset. Define its $\zeta$-function as

$$\zeta_{\mathcal{P}}(x, y) \triangleq [x \preccurlyeq y] = \begin{cases} 1 & \text{if } x \preccurlyeq y \\ 0 & \text{otherwise} \end{cases},$$

and define Möbius function as the inverse of $\zeta_{\mathcal{P}}$: $\mu_{\mathcal{P}} \triangleq \zeta_{\mathcal{P}}^{-1}$. Now we have the Möbius inversion formula for posets.

**Theorem 4.20** (Möbius inversion formula)**.** *Let $\mathcal{P} = (P, \preccurlyeq)$ be a poset, and $f, g : P \to \mathbb{R}$ be two functions. Suppose*

$$f(x) = \sum_{x \preccurlyeq y} g(y).$$

*Then it holds that*

$$g(x) = \sum_{x \preccurlyeq y} \mu(x, y) f(y).$$

*Similarly, if*

$$f(x) = \sum_{y \preccurlyeq x} g(y),$$

*then we have*

$$g(x) = \sum_{y \preccurlyeq x} \mu(y, x) f(y).$$

*Proof.* If $f = \zeta g$, then $\mu f = \mu\zeta g = g$. Similarly, if $f = \zeta^{\intercal} g$, then $\mu^{\intercal} f = \mu^{\intercal}\zeta^{\intercal} g = (\zeta\mu)^{\intercal} g = g$. $\square$

## 4.4 Möbius functions and examples

We now show that Möbius functions exist. By definition, we have

$$\forall x, y \in P, \quad \sum_{z \in [x,y]} \mu(x, z) = \sum_{z \in [x,y]} \mu(z, y) = \delta(x, y). \tag{4.1}$$

It is easy to see that $\mu(x, y) = 1$ if $x = y$, and $\mu(x, y) = -1$ if $x \prec y$. Then $\mu$ can be inductively determined by $|[x, y]|$ as follows:

$$\mu(x, y) = \begin{cases} 0 & \text{if } x \not\preccurlyeq y \\ 1 & \text{if } x = y \\ \sum\limits_{x \preccurlyeq z \prec y} -\mu(x, z) & \text{o.w.} \end{cases} = \begin{cases} 0 & \text{if } x \not\preccurlyeq y \\ 1 & \text{if } x = y \\ \sum\limits_{x \prec z \preccurlyeq y} -\mu(z, y) & \text{o.w.} \end{cases}.$$

We can verify that $\mu$ is the inverse of $\zeta$ directly:

$$(\mu\zeta)(x,y) = \sum_{x \preccurlyeq z \preccurlyeq y} \mu(x,z) = \begin{cases} 0 & \text{if } x \not\preccurlyeq y\,, \\ 1 & \text{if } x = y\,, \\ \mu(x,y) + \sum_{x \preccurlyeq z \prec y} \mu(x,z) = 0 & \text{if } x < y\,. \end{cases}$$

**Example 4.21.** Let $\mathcal{P} = ([n], \leq)$. The Möbius function is

$$\mu(x,x) = 1\,,$$
$$\mu(x,y) = -\sum_{x \leq z < y} \mu(x,z) = \begin{cases} -1 & \text{if } y = x+1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Now we consider some more complicated posets, e.g., $(2^{[n]}, \subseteq)$. We need the following lemma.

**Definition 4.22.** Let $\mathcal{P} = (P, \preccurlyeq_P)$, $\mathcal{Q} = (Q, \preccurlyeq_Q)$ be two posets. Define their product as

$$\mathcal{P} \times \mathcal{Q} \triangleq (P \times Q, \preccurlyeq)\,,$$

where $(x_1, x_2) \preccurlyeq (y_1, y_2)$ iff $x_1 \preccurlyeq_P y_1$ and $x_2 \preccurlyeq_Q y_2$.

**Lemma 4.23.** *Let $\mathcal{P} = (P, \preccurlyeq_P)$, $\mathcal{Q} = (Q, \preccurlyeq_Q)$ be two posets, and $\mu_P, \mu_Q$ be their Möbius functions. The Möbius function of their product $\mathcal{P} \times \mathcal{Q}$ satisfies*

$$\mu\big((x_1,x_2),(y_1,y_2)\big) = \mu_P(x_1,y_1) \cdot \mu_Q(x_2,y_2)\,. \tag{4.2}$$

*Proof.* We prove it by induction on $|[(x_1,x_2),(y_1,y_2)]|$. If the size is 0 or 1, it is trivial. Suppose $|[(x_1,x_2),(y_1,y_2)]| \geq 2$ and equation (4.2) holds for smaller $|[(x_1,x_2),(y_1,y_2)]|$. We verify equation (4.2) by applying (4.1). Note that now we have $x_1 \neq y_1$ or $x_2 \neq y_2$, so either $\sum_{z_1 \in [x_1,y_1]} \mu_P(x_1,z_1) = 0$, or $\sum_{z_2 \in [x_2,y_2]} \mu_Q(x_2,z_2) = 0$. Thus,

$$\begin{aligned} \mu(\boldsymbol{x},\boldsymbol{y}) &= 0 - \sum_{\boldsymbol{x} \preccurlyeq \boldsymbol{z} \prec \boldsymbol{y}} \mu(\boldsymbol{x},\boldsymbol{z}) \\ &= \Bigg(\sum_{x_1 \preccurlyeq z_1 \preccurlyeq y_1} \mu_P(x_1,z_1)\Bigg)\Bigg(\sum_{x_2 \preccurlyeq z_2 \preccurlyeq y_2} \mu_Q(x_2,z_2)\Bigg) - \sum_{\boldsymbol{x} \preccurlyeq \boldsymbol{z} \prec \boldsymbol{y}} \mu(\boldsymbol{x},\boldsymbol{z}) \\ &= \sum_{\boldsymbol{x} \preccurlyeq \boldsymbol{z} \preccurlyeq \boldsymbol{y}} \mu_P(x_1,z_1) \cdot \mu_Q(x_2,z_2) - \sum_{\boldsymbol{x} \preccurlyeq \boldsymbol{z} \prec \boldsymbol{y}} \mu_P(x_1,z_1) \cdot \mu_Q(x_2,z_2) \\ &= \mu_P(x_1,y_1) \cdot \mu_Q(x_2,y_2)\,, \end{aligned}$$

where we denote $\boldsymbol{x} = (x_1,x_2), \boldsymbol{y} = (y_1,y_2)$ and $\boldsymbol{z} = (z_1,z_2)$. □

**Example 4.24.** It is easy to calculate the Möbius function of $\mathcal{P} = (2^{[n]}, \subseteq)$ by Lemma 4.23. Note that $\mathcal{P}$ is isomorphic to $\big(\{0,1\}, \leq\big)^n$. So we have $\mu(S,T) = (-1)^{|T \setminus S|}$ if $S \subseteq T$.

**Example 4.25.** Let $D_n$ be the set of divisors of $n$. We can calculate the Möbius function of $\mathcal{P} = (D_n, |)$ by Lemma 4.23. Suppose $n = p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}$ for distinct primes $p_1, \ldots, p_t$. Then $\mathcal{P}$ is isomorphic to

$$([r_1 + 1], \leq) \times ([r_2 + 1], \leq) \times \cdots \times ([r_t + 1], \leq).$$

Thus, we have

$$\mu(a,b) = \begin{cases} 1 & \text{if } a = b\,, \\ 0 & \text{if } a \nmid b \text{, or } p_i^2 \mid \frac{b}{a} \text{ for some } p_i\,, \\ (-1)^k & \text{if } \frac{b}{a} \text{ is the product of } k \text{ distinct primes}\,. \end{cases}$$

Note that nonzero values of $\mu(a,b)$ only depend on $\frac{b}{a}$, so we can use the notation $\mu(\frac{b}{a}) = \mu(a,b)$, which is the same as the number-theoretical Möbius function. Then the Möbius inversion formulas on $\mathcal{P}$ are also the same as number-theoretical Möbius inversion.

In fact, $(D_n, |)$ is a lattice. For such a finite lattice, we can also apply the following theorem.

**Theorem 4.26** (Weisner's theorem)**.** *Let $\mathcal{L}$ be a finite lattice, and $\mu$ be its Möbius function. Suppose $\hat{0}$ and $\hat{1}$ are the minimum and maximum elements in $\mathcal{L}$. Then, for all $y \neq \hat{0}$,*

$$\sum_{x : x \vee y = \hat{1}} \mu(\hat{0}, x) = 0\,.$$

*Proof.* Applying (4.1), we have

$$\begin{aligned} \sum_{x : x \vee y = \hat{1}} \mu(\hat{0}, x) &= \sum_{x} \mu(\hat{0}, x)\, \delta(x \vee y, \hat{1}) \\ &= \sum_{x} \mu(\hat{0}, x) \sum_{(x \vee y) \preccurlyeq z \preccurlyeq 1} \mu(z, \hat{1}) \\ &= \sum_{y \preccurlyeq z} \mu(z, \hat{1}) \sum_{x \preccurlyeq z} \mu(\hat{0}, x) \\ &= \sum_{y \preccurlyeq z} \mu(z, \hat{1})\, \delta(\hat{0}, z) = 0\,. \end{aligned}$$

□

**Example 4.27.** Let $\mathcal{L} = (2^{[n]}, \subseteq)$. Then for any $T \neq \emptyset$, we have $\sum_{S : S \cup T = [n]} \mu(\emptyset, S) = 0$. Thus by setting $T$ to be a singleton set, it yields

$$\mu(\emptyset, S) = -\mu(\emptyset, [n])$$

if $|S| = n - 1$. Similarly, by induction we obtain

$$\mu(\emptyset, S) = (-1)^{n-|S|} \mu(\emptyset, [n])\,.$$

Note that $\mu(\emptyset, S) = -1$ if $|S| = 1$. Hence, we conclude that $\mu(\emptyset, S) = (-1)^{|S|}$.

**Exercise 4.28.** *Calculate $\mu(1, n)$ in $\mathcal{L} = (D_n, |)$ by Theorem 4.26.*