

5

Graph Theory Basics

In most cases, a graph is just a relation on a set. It abstracts a set of objects and their pairwise relations.

5.1 Basic definitions and concepts

We define a graph G as an ordered pair (V, E) with an incidence function. Unless otherwise stated, V and E are finite. Usually, we only consider simple graphs, which are graphs without parallel edges and without self-loops.

Definition 5.1 (Graph). A simple graph is a pair $G = (V, E)$, where V is a finite set (a.k.a. the *vertex set*), and $E \subseteq \binom{V}{2}$ (a.k.a. the *edge set*). For directed graphs (a.k.a. *digraphs*), E is viewed as a subset of $V \times V$.

If not specified, we usually use $V(G)$ to denote the vertex set of G , and $E(G)$ to denote the edge set. It is standard to write $v(G) \triangleq |V(G)|$ and $e(G) \triangleq |E(G)|$. It is also conventional to write them as n and m . They are called the *order* and *size* of the graph, respectively.

We write $uv \in E$ (or $u \sim v$, $\{u, v\} \in E$) to indicate that there is an edge between u and v in G . In this case u and v are said to be *adjacent*, and they are *incident* to the edge. For directed graphs, we write $\vec{uv} \in E$ (or $u \rightarrow v$, $(u, v) \in E$).

An *oriented graph* is a simple graph where we pick a direction for each edge. Clearly each simple graph can be oriented in 2^m ways.

Example 5.2. There are $2^{\binom{n}{2}}$ simple graphs on n vertices, 2^{n^2} digraphs, and $3^{\binom{n}{2}}$ oriented graphs.

Definition 5.3 (Neighbor and degree). In a (simple) graph, the *neighbors* of the vertex v is the set $N(v) \triangleq \{u \mid uv \in E\}$. The *degree* of v is $d(v) \triangleq |N(v)|$.

We can also define neighbors and degrees on a set of vertices, that is, $N(S) \triangleq \cup_{v \in S} N(v)$ and $d(S) \triangleq |N(S)|$.

The smallest and the largest degree in G are denoted $\delta(G)$ and $\Delta(G)$, respectively.

In a graph with parallel edges or self-loops, $d(v)$ is defined as the number of edges incident to v .

In a digraph, N^- , N^+ , d^- (in-degree), d^+ (out-degree), δ^- , δ^+ , Δ^- , Δ^+ can be similarly defined, where $-$ means edges going into a vertex, and $+$ means edges going out.

By counting the number of “vertex v incident to edge e ” relations in two ways, we get the following theorem, which is often the first theorem one learns in graph theory.

Theorem 5.4 (Handshaking lemma). *In any graph, $\sum_{v \in V} d(v) = 2|E|$. As a result, the number of vertices with odd degree is even.*

This result looks very simple, but it can give some surprising corollaries.

Theorem 5.5 (Sperner’s lemma). *Subdivide $\triangle ABC$ into a triangulation (small triangles meet edge-by-edge) and color the vertices by A, B, C . Each vertex that lies along any edge of $\triangle ABC$ can be only colored with the two colors of the endpoints of the edge, where the interior vertices can be colored arbitrarily. Then, there exists a small “tri-colored” triangles.*

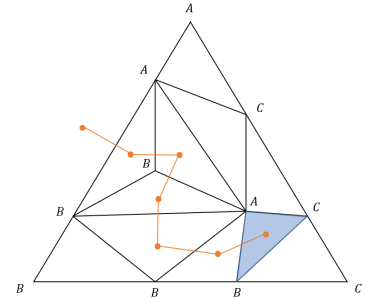
Proof. Consider the dual graph whose vertices are the outer part and all small pieces of triangles, and vertices are only connected by crossing A - B edges. Notice that the outside vertex has odd degree, which implies that there exists a vertex with odd degree inside $\triangle ABC$. It is easy to see that only those vertices corresponding to “tri-colored” triangles have odd degrees, which completes the proof. $\square$

Similar to Theorem 5.4, for any digraph, we have

$$\sum_{v \in V} d^+(v) = \sum_{v \in V} d^-(v) = |E|.$$

Definition 5.6. A undirected graph is k -regular if $d(v) = k$ for all $v \in V$. A directed graph is k -diregular if $d^+(v) = d^-(v) = k$ for all v .

Example 5.7. A complete graph of order n , denoted K_n , is the graph where all $\binom{n}{2}$ edges are present. K_n is $(n-1)$ -regular.



The picture shows a triangulation of $\triangle ABC$. The blue triangle is a small “tri-colored” triangle. The orange vertices and edges show the dual graph.

Sperner’s lemma can be generalized to n -dimensional simplices. Although it asserts that there always exists a sub-simplex whose different vertices are colored with different colors, it is difficult to find such a sub-simplex. Note that the decision problem cannot be NP-hard since the answer is always “yes”. In fact, Sperner’s lemma is another form of Brouwer’s fixed point, and finding an n -colored sub-simplex for a given Sperner coloring is PPAD-complete.

An orientation for the complete graph is called a *tournament*. For a labeled graph of order n , there are $2^{\binom{n}{2}}$ tournaments, which can be viewed as all the possible outcomes of a round robin tournament with n teams. A tournament may be not diregular.

Definition 5.8 (Isomorphism and automorphism). A graph H is *isomorphic* to a graph G , denoted $H \cong G$, if there is a bijection f between $V(H)$ and $V(G)$, such that $uv \in E(H)$ iff $f(u)f(v) \in E(G)$, where f is called an *isomorphism*.

An *automorphism* of G is an *isomorphism* from G to itself. It is easy to check that the set of automorphisms, denoted $\text{Aut}(G)$, form a subgroup of the symmetric group on V .

A graph is *vertex transitive* if for any two vertices u and v , there is an automorphism that maps u to v .

Example 5.9. Here are three isomorphic copies of the *Petersen graph*, where $V = \binom{[5]}{2}$, and $\{a, b\} \sim \{c, d\}$ iff a, b, c, d are distinct.

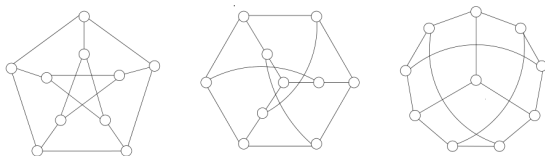


Figure 5.1: Petersen graph

Example 5.10. If a graph is vertex transitive, it is k -regular for some k . If a digraph is vertex transitive, it is k -diregular for some k .

The isomorphic relation is an equivalence relation thus it gives a partition of all the graphs. Graphs isomorphic to each other share the same graph properties.

5.2 Walks, paths, cycles and connectivity

Definition 5.11 (Walk, path and cycle). A *walk* of length k in a graph is a sequence of vertices $u_0, u_1, \dots, u_k$ where $u_i \sim u_{i+1}$. If $u_0 = u_k$, it is called a *closed walk*.

A *path* of length k is a walk of length k consisting of distinct vertices.

If $k \geq 3$ and all the vertices are different except $u_0 = u_k$, it is called a *cycle*.

P_n is the graph on n vertices which form an $(n - 1)$ -path.

C_n is the graph on n vertices which form an n -cycle.

Example 5.12. P_3 is the graph of 2 edges sharing a common endpoint. C_3 is a triangle.

Definition 5.13 (Girth). The *girth* of a graph is the length of the shortest cycle in the graph.

Example 5.14. A graph with girth 5 and $\delta = d$ has at least $d^2 + 1$ vertices.

Definition 5.15 (Complement). The *complement* of a (simple) graph $G = (V, E)$, denoted $\overline{G}$, is $\overline{G} = (V, \binom{V}{2} \setminus E)$.

Example 5.16. The complement of K_n is the *empty graph* of order n . The complement of P_4 is also a P_4 . The complement of C_5 is also a C_5 .

Definition 5.17 (Subgraph). A *subgraph* of a (simple) graph $G = (V, E)$ is a pair $G' = (V', E')$, where $V' \subseteq V$, $E' \subseteq E \cap \binom{V'}{2}$. G' is an *induced subgraph*, usually denoted $G[V']$, if $E' = E \cap \binom{V'}{2}$, that is, for each pair of u, v , if $u, v \in V'$ and $u \sim v$ in G , then $u \sim v$ in G' . G' is a *spanning subgraph* if $V' = V$.

Definition 5.18. A graph is *connected* if there is a path between each pair of vertices. Otherwise it is *disconnected*.

A *connected component* of a graph is a maximal connected subgraph.

Proposition 5.19. If $G = (V, E)$ is disconnected, then we can find a partition $V = A \uplus B$ such that there are no edges between A and B .

Proof. Fix a vertex $v \in V$. Let A be the set of vertices reachable from v , and $B = V \setminus A$. □

If G is not connected, there is a partition of vertices $V = A \uplus B$ where $\overline{G}$ contains all the edges between A and B , and thus connected. So we obtain the following fact.

Proposition 5.20. Either G or its complement is connected.

Proposition 5.21. If G is disconnected, there are at most $\binom{n-1}{2}$ edges in G .

5.3 Eulerian and Hamiltonian

The first paper of the graph theory was written by L. Euler on the problem of the bridges of Königsberg (formerly a city in Prussia, now

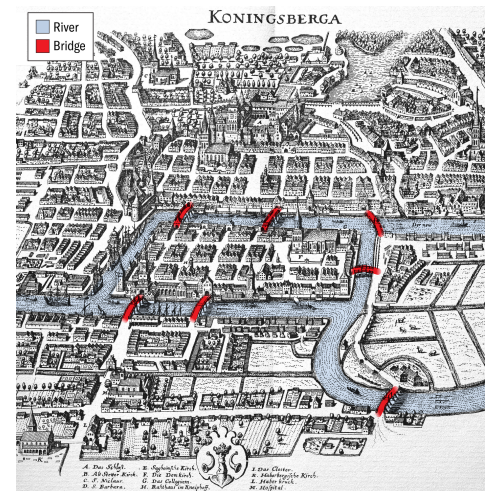


Figure 5.2: Seven bridges in Königsberg

Kaliningrad in Russia): is it possible to make a walk through the city, returning to the starting point and crossing each bridge exactly once? This paper has led to the following definition.

Definition 5.22 (Eulerian). An *Eulerian walk*, a.k.a. *Eulerian trail*, or *Eulerian path*, is a walk on a graph which uses every edge exactly once.

An *Eulerian circuit*, a.k.a. *Eulerian cycle*, or *Eulerian tour*, is a closed Eulerian walk. A graph that has such a circuit is called *Eulerian*.

Theorem 5.23. A finite graph G (with possible parallel edges) is Eulerian iff it is connected and each vertex has even degree.

Proof sketch. The necessity is trivial. To prove the sufficiency, we start in a vertex v and begin making a path. Keep going, never using the same edge twice, until we cannot go further. Since every vertex has even degree, this can only happen when we return to v and all edges from v have been used. If all edges are used, then we're done. Otherwise choose a visited vertex that is incident to an unused edge, and start the procedure again. Finally, these walks can be combined to a longer closed walks. $\square$

Such a vertex must exist, since the graph is connected.

It is easy to obtain the following corollary.

Corollary 5.24. A finite graph G (with possible parallel edges) has an Eulerian walk iff it is connected and at most two vertices has odd degree.

The same result can be developed in diagraphs. A diagraph is Eulerian iff it is connected and $d^+(v) = d^-(v)$ for all v .

Corollary 5.25. Every simple graph G has a balanced orientation, where an oriented graph is called balanced if for each vertex v ,

$$|d^+(v) - d^-(v)| \leq 1.$$

Proof. We may assume the graph is connected. Otherwise consider each connected components separately. Since the number of vertices with odd degree is even, we pair such vertices arbitrarily, and add an (imaginary) edge between each pair. Now we obtain an Eulerian graph, and orient the edges according to an Eulerian tour. So every vertex has the same in-degree and out-degree. Finally we remove all imaginary edges, and each vertex has at most one edge removed. $\square$

If we require the walk passing through each vertex exactly once rather than each edge, the walk is called *Hamiltonian*.

Definition 5.26 (Hamiltonian). A *Hamiltonian path* in a graph G is a simple path that passes through each vertex exactly once. A *Hamiltonian circuit* is a closed Hamiltonian path. A graph that has such a circuit is called *Hamiltonian*.

Example 5.27. A graph admits a Hamiltonian circuit if and only if it has a polygon as a spanning subgraph.

Example 5.28. Petersen graph is not Hamiltonian.

As we see, deciding Eulerian properties is easy. However, deciding Hamiltonian property is NP-complete.

Theorem 5.29 (Dirac, 1952). If G is a graph on $n \geq 3$ vertices and $\delta \geq n/2$, then G is Hamiltonian.

It can be clearly derived from the following theorem.

Theorem 5.30 (Ore, 1960). If G is a graph on $n \geq 3$ vertices and $d_u + d_v \geq n$ whenever $u \not\sim v$, then G is Hamiltonian.

This theorem is a direct corollary of the following one.

Theorem 5.31 (Bondy-Chvátal, 1972). If $G = (V, E)$ is a graph on $n \geq 3$ vertices, $u, v \in V$ and $d_u + d_v \geq n$, then G is Hamiltonian iff G' is, where $G' = (V, E \cup \{uv\})$.

Proof. The " $G \implies G'$ " direction is trivial. Now assume $u \not\sim v$ in G , and G' has a Hamiltonian circuit that uses uv : $u, v, w_1, \dots, w_{n-2}$. So u has $d_u - 1$ neighbors in $\{w_1, \dots, w_{n-3}\}$. Since $d_u + d_v \geq n$, there exists $1 \leq k \leq n - 3$ such that $u \sim w_k$ and $v \sim w_{k+1}$. Thus, $v, w_1, \dots, w_k, u, w_{n-2}, \dots, w_{k+1}$ is a Hamiltonian circuit in G . $\square$

Exercise 5.32. Any tournament has a Hamiltonian path.

5.4 Trees and Cayley's formula

Definition 5.33 (Tree and forest). A *tree* is a connected graph that contains no cycles.

A *forest* is a graph with no cycles as subgraphs, that is, every connected component is a tree.

Proposition 5.34. *In a graph G on n vertices, any two of the following three conditions imply the third one:*

1. G is connected;
2. G has no cycle;
3. $e(G) = n - 1$.

Definition 5.35. A vertex with degree 1 in a tree is called a *leaf*.

Proposition 5.36. *In a tree of order at least 2, there are at least two leaves.*

Proof. The sum of degrees is $2n - 2$. □

Exercise 5.37. Let $A_1, \dots, A_n$ be n distinct subsets of $[n]$. Show that there is a $v \in [n]$ such that the sets $A_i \setminus \{v\}$ are all distinct.

Proof. Let G be the graph on $A_1, A_2, \dots, A_n$ that add an edge labeled v between A_i and A_j iff $A_i \Delta A_j = \{v\}$. Suppose G is not a forest. Choose any cycle in G , and any edge e on the cycle. Let v be the label of e . Note that each label must appear even times on the cycle. So there is another edge labeled u on the cycle. We can remove e from G so that the number of distinct labels does not change. Remove such edges again and again until G is a forest. Now there are at most $n - 1$ distinct labels, and let $v \in [n]$ be any unused number. □

The symmetric difference of two sets S and T , denoted $S \Delta T$, is defined as

$$S \Delta T \triangleq (S \setminus T) \cup (T \setminus S).$$

Definition 5.38 (Spanning tree). A *spanning tree* of a graph G is a spanning subgraph of G that is a tree.

The following formula is one of the most well-known results on spanning trees, or labeled trees.

Theorem 5.39 (Cayley's formula, 1889). *There are n^{n-2} different spanning trees in K_n .*

Equivalently, the Cayley's formula states that the number of all labeled trees on $[n]$ is n^{n-2} . The term "labeled" emphasizes that we are not identifying isomorphic graphs. Two trees are counted as the same iff exactly the same pairs of vertices are adjacent.

There are a variety of elegant proofs on Cayley's formula, for example, see Chapter 33, *Proofs from THE BOOK*. Here we introduce the (possibly) most well-known one, due to H. Prüfer.

Proof. We introduce the *Prüfer code*, which gives a bijection between all labeled trees on $[n]$ and all $(n - 2)$ -sequences on $[n]$.

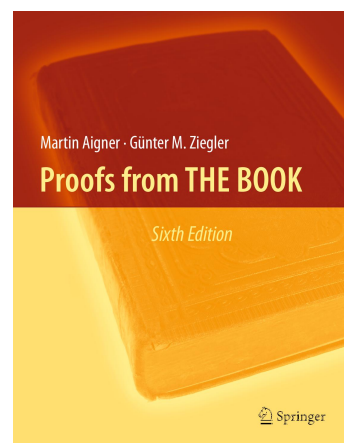


Figure 5.3: Proofs from THE BOOK

Given a labeled tree T on $[n]$, find the leaf v with the smallest label. Then write down the label of its neighbor and delete v (and its incident edge) from T . Repeat this procedure until there are only two vertices remaining in T .

Next, we show that this is a bijection. To see this, we claim that given any $(n-2)$ -sequence $(v_1, \dots, v_{n-2}) \in [n]^{n-2}$, we can uniquely decode one tree. Let $v_{n-1} = n$, and we construct $u_1, \dots, u_{n-1}$ inductively: u_k is the smallest number in $[n]$ that does not appear in $\{u_1, \dots, u_{k-1}\} \cup \{v_k, \dots, v_{n-1}\}$. Note that the set has size $n-1$, so u_k always exists. Now we can prove that $T = ([n], E)$ is the desired tree, where $E = \{\{u_1, v_1\}, \dots, \{u_{n-1}, v_{n-1}\}\}$.

In fact, we can show $(u_1, \dots, u_{n-1})$ is the list of deleted vertex at each step. By Proposition 5.36, vertex n cannot be deleted at any time. So n should not be in $\{u_1, \dots, u_{n-1}\}$. Then we can see that the number of appearance of k in $\{v_1, \dots, v_{n-2}\}$ is $d(k) - 1$, that is because k appears exactly once in $\{u_1, \dots, u_{n-1}\}$ for all $k \in [n-1]$, and $v_{n-1} = n$. So any subset of E contains a degree-1 vertex, which, combining with $|E| = n-1$, implies that T is a tree. Similarly, let $V_k = \{u_k, \dots, u_{n-1}, n\}$, $E_k = \{\{u_k, v_k\}, \dots, \{u_{n-1}, v_{n-1}\}\}$ and $T_k = (V_k, E_k)$. Then T_k is a tree. For each T_k , by the rule to construct Prüfer code, we need to find the smallest leaf in T_k , which is exactly u_k , since u_k is the smallest one that does not appear in $\{v_k, \dots, v_{n-1}\}$ (i.e., has degree 1 in T_k). Clearly, we will delete vertex u_k and edge $\{u_k, v_k\}$ from T_k (then we obtain T_{k+1}), and write down v_k . Inductively we prove that $(v_1, \dots, v_{n-2})$ is the Prüfer code of such T . $\square$

By the proof of Prüfer code, we obtain the following corollary immediately.

Corollary 5.40. *The number of labeled trees with the given degree sequence $(d_1, \dots, d_n)$ is*

$$\binom{n-2}{d_1-1, \dots, d_n-1} = \frac{(n-2)!}{(d_1-1)! \cdots (d_n-1)!}.$$

Actually, this result can also be derived from recurrence relation. Let $t(n; d_1, \dots, d_n)$ be the number of labeled trees with given degree sequence. Note that the number remains the same on any permutation of $d_1, \dots, d_n$. Assume $d_n = 1$ w.l.o.g. Then we obtain

$$t(n; d_1, \dots, d_n) = \sum_{i=1}^{n-1} t(n-1; d_1, \dots, d_{i-1}, d_i-1, d_{i+1}, \dots, d_{n-1})$$

by enumerating the neighbor of n . Comparing with multinomials

$$\binom{n}{r_1, \dots, r_m} = \sum_{i=1}^m \binom{n-1}{r_1, \dots, r_{i-1}, r_i-1, r_{i+1}, \dots, r_m},$$

Corollary 5.40 also can be concluded by induction.

5.5 Planar graphs

Definition 5.41 (Planar graph). A graph is *planar* if it admits a drawing on the plane such that the edges do not cross but meet only at vertices. We talk of a *plane graph* if such a drawing is already given and fixed.

Any such drawing decomposes the plane or sphere into a finite number of connected regions, including the outer (unbounded) region, which are referred to as *faces*. The following Euler's formula exhibits a beautiful relation between the number of vertices, edges and faces that is valid for any plane graph.

Theorem 5.42 (Euler's formula). If G is a connected plane graph with v vertices, e edges and f faces, then

$$v - e + f = 2.$$

Proof. We introduce the *dual graph* G^* of G , where the vertex set is the set of all faces, and each pair of faces are connected by an edge in G^* iff they share a common boundary edge in G . (If there are several common boundary edges, then we draw several connecting edges in the dual graph.)

Let $T \subseteq E$ be the edge set of a spanning tree for G , and let $T^* \subseteq E^*$ be the set of edges in the dual graph that corresponds to edges in $E \setminus T$. The edges in T^* connect all the faces, since T does not have a cycle; but also T^* does not contain a cycle, since otherwise it would separate some vertices of G inside the cycle from vertices outside. Thus T^* is a spanning tree for G^* . $\square$

Since each face has at least 3 boundary edges, and each edge is the boundary of 2 faces, we have $2e \geq 3f$.

Corollary 5.43. Let G be any simple planar graph on n vertices. Then

1. G has at most $3n - 6$ edges;
2. G has a vertex with degree at most 5.

Example 5.44. K_5 is not planar.

Another important example of non-planar graph is the complete bipartite graph $K_{3,3}$.

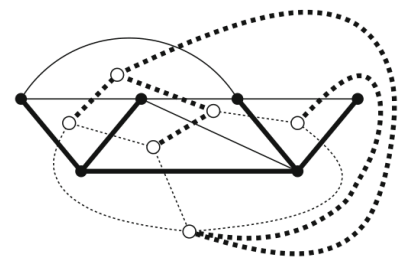


Figure 5.4: Dual spanning trees in G and in G^*

Definition 5.45 (Bipartite graph). A graph $G = (V, E)$ is *bipartite* if V can be partitioned into A and B such that all the edges are between A and B . Similarly, we can define *k-partite* graphs.

Example 5.46. Trees are bipartite graphs.

Example 5.47. The *complete bipartite graph* $K_{m,n}$ is the graph with m vertices in one part, n vertices in the other, and all the mn possible edges. $K_{1,n}$ is called a *star*. In particular, $K_{1,3}$ is also known as a *claw*.

Note that $K_{3,3}$ has 6 vertices and 9 edges that does not violate Corollary 5.43. But it is not planar yet, since Corollary 5.43 is not tight for bipartite graphs.

Proposition 5.48. A graph is bipartite iff it contains no odd cycles.

Corollary 5.49. Any simple bipartite planar graph on n vertices has at most $2n - 4$ edges.

In fact, K_5 and $K_{3,3}$ are two fundamental structure for non-planar graphs. Of course, any planar graph cannot have K_5 or $K_{3,3}$ as its subgraph, but not vice versa (e.g., the Petersen graph).

Definition 5.50 (Subdivision). An *edge subdivision* is the insertion of a new vertex into an existing edge. A *graph subdivision* is a sequence of edge subdivisions.

Theorem 5.51 (Kuratowski's theorem). A graph is planar if and only if it does not contain subgraphs isomorphic to subdivisions of K_5 or $K_{3,3}$.

Example 5.52. The Petersen graph has a subdivision of $K_{3,3}$.

Definition 5.53. In a graph $G = (V, E)$, *deletion* of an edge $e = \{u, v\}$, denoted $G - e$, is the subgraph $(V, E \setminus e)$; and *contraction* of e , denoted $G \setminus e$, is the graph obtained by replacing u and v with a single vertex such that edges incident to the new vertex are the edges other than e that were incident to u or v .

Definition 5.54 (Graph minor). A graph H is a *minor* of a graph G if H can be obtained from G via repeatedly deleting and contracting edges (and possibly deleting some or all isolated vertices).

Theorem 5.55 (Wagner's theorem). *A graph is planar iff it does not contain K_5 or $K_{3,3}$ as a graph minor.*

Example 5.56. The Petersen graph has both K_5 and $K_{3,3}$ as its minor.

5.6 Graph colorings

Perhaps the most well-known theorem on planar graphs is the *Four-color theorem*. We now introduce *graph colorings*.

Definition 5.57 (Graph coloring). A *proper vertex coloring* of a graph G is an assignment of a set C of “colors” to each vertex of G such that no edge connects two identically colored vertices. If $|C| = k$, it is called a *k -coloring*, and G is said to be *k -colored*. Similarly we can define *proper edge colorings*, where no vertex is incident to two identically colored edges.

Unless specified, a graph coloring usually means a vertex coloring.

Definition 5.58 (Chromatic number). The *chromatic number* of a graph G , denoted $\chi(G)$, is the minimum number of colors for which a proper coloring exists.

Example 5.59. $\chi(G) = 1$ iff G is empty. $\chi(G) = 2$ iff G is nonempty and bipartite. $\chi(K_n) = n$.

Theorem 5.60 (Four-color theorem). *If G is planar, then $\chi(G) \leq 4$.*

Theorem 5.61. *Let $d \geq 3$, and G be a graph with maximum degree $\Delta \leq d$. If G does not contain K_{d+1} as a subgraph. Then $\chi(G) \leq d$.*

Remark 5.62. This is **NOT TRUE** for $d = 2$, e.g., odd cycles.

In fact, “ K_t as a subgraph” is not believed to be a good characterization of “ t -colorable”.

Proof. We prove by contradiction. Suppose this is not true for some $d \geq 3$, and let H be a minimal graph (with respect to the number of vertices) with $\Delta(H) \leq d$, $\chi(H) > d$, and $K_{d+1} \not\subseteq H$.

Since H is not complete, there exists three vertices v_1, v_{n-1}, v_n such that $v_1 \sim v_{n-1}$, $v_1 \sim v_n$ but $v_{n-1} \not\sim v_n$. Now we can list the vertices in H in the order

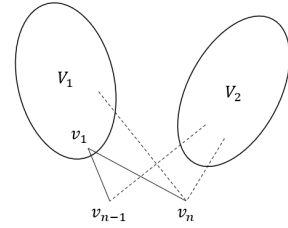
$$v_1, v_2, \dots, v_{n-2}, v_{n-1}, v_n$$

such that each v_{i+1} is adjacent to at least one of $v_1, \dots, v_i$ for all $1 \leq i \leq n-3$. If it is not possible, say, no vertex except v_{n-1} and v_n is adjacent to at least one of $v_1, \dots, v_k$ for some $1 \leq k \leq n-3$, then we can show that H is not a minimal graph that cannot be d -colored. Let $V_1 = \{v_1, \dots, v_k\}$ and $V_2 = (V \setminus V_1) \setminus \{v_{n-1}, v_n\}$. So there is no edge between V_1 and V_2 . Since H is minimal, H is connected and *not separable* (i.e., every graph obtained from H by deleting a vertex is connected). It follows that both v_{n-1} and v_n are adjacent to V_2 , and at least one of v_{n-1} and v_n is adjacent to $V_1 \setminus \{v_1\}$.

Consider two subgraphs $H_1 = H[V_1 \cup \{v_{n-1}, v_n\}]$ and $H_2 = H[V_2 \cup \{v_{n-1}, v_n\}]$. By assumption, H_1 and H_2 are d -colorable. Let $c_1 : V_1 \cup \{v_{n-1}, v_n\} \rightarrow [d]$ be a coloring of H_1 such that $c_1(v_{n-1}) \neq c_1(v_n)$. If such a coloring does not exist, both v_{n-1} and v_n have degree $d-1$ in V_1 . Thus both have degree 1 in V_2 , and it is easy to find a coloring $c_2 : V_2 \cup \{v_{n-1}, v_n\} \rightarrow [d]$ of H_2 such that $c_2(v_{n-1}) = c_2(v_n) = d$. Then choose any coloring c_1 such that $c_1(v_{n-1}) = c_1(v_n) = d$. Clearly $c_1 \cup c_2$ is a proper coloring of H . Otherwise, note that $H_2 \cup \{v_{n-1}, v_n\}$ is still a graph with $\Delta \leq d$ and without K_{d+1} as a subgraph, and thus is d -colorable. So there is a coloring c_2 of H_2 such that $c_2(v_{n-1}) \neq c_2(v_n)$. Permute d colors in c_2 so that $c_2(v_{n-1}) = c_1(v_{n-1})$ and $c_2(v_n) = c_1(v_n)$. Then $c_1 \cup c_2$ is a proper coloring of H .

Hence we obtain a list of vertices in H in the order $v_1, \dots, v_n$ such that each v_{i+1} is adjacent to at least one of $v_1, \dots, v_i$. Now we d -color H from v_n to v_1 , which leads to a contradiction. Since $v_{n-1} \approx v_n$, assign them the same color. Then for each v_k ($2 \leq k \leq n-2$), it is adjacent to at most $d-1$ vertices in $\{v_{k+1}, \dots, v_n\}$. So there is an available color for v_k . Finally, there is a color available for v_1 , since it has two neighbors v_{n-1} and v_n that are given the same color. $\square$

Why?



Conjecture 5.63 (Hadwiger's conjecture). Every loopless graph with no K_{t+1} -minor is t -colorable.

We now consider the lower bound of $\chi(G)$. It is easy to see that the set of vertices with the same color forms an *independent set*.

Definition 5.64. In a graph, an *independent set* is a set of vertices in which any two vertices are not adjacent; a *clique* is a set of vertices in which any two vertices are adjacent.

The size of the *maximum independent set* in a graph G is denoted $\alpha(G)$, and the size of the *maximum clique* is denoted $\omega(G)$.

Example 5.65. An independent set in a graph G is a clique in $\overline{G}$. So $\alpha(G) = \omega(\overline{G})$.

Theorem 5.66. In a graph on n vertices, $\alpha(G) \geq n/\chi(G)$.

Now we turn to count the number of proper k -colorings.

Example 5.67. There are $k(k-1)^{n-1}$ k -colorings of a tree on $[n]$.

The number of proper colorings can be computed by the principle of inclusion and exclusion. We consider all mappings from V to $[k]$, and define bad events as $B_{u,v} = \{\sigma : V \rightarrow [k] \mid \sigma(u) = \sigma(v)\}$ for all $\{u, v\} \in E$. It is easy to compute $\cap_{e \in M} B_e$.

Definition 5.68. Let $G = (V, E)$ be a graph. There is a degree n polynomial $\chi_G(\lambda)$, such that for any positive integer λ , $\chi_G(\lambda)$ gives the number of proper λ -colorings of G . $\chi_G(\lambda)$ is called the *chromatic polynomial*.

Example 5.69. Let G be a graph on n vertices and m edges, and

$$\chi_G(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0.$$

Then $a_n = 1$, $a_0 = 0$, $a_{n-1} = -m$ and $a_{n-2} = \binom{m}{2} - t$, where t is the number of triangles in G .

Theorem 5.70 (June Huh, 2012). *The coefficients of a chromatic polynomial is log-concave.*

Recall Definition 5.53 that we use $G - e$ to denote the subgraph of deleting e and use $G \setminus e$ to denote the subgraph of contracting e . Then the chromatic polynomial has a recurrence relation.

Theorem 5.71. $\chi_G(\lambda) = \chi_{G-e}(\lambda) - \chi_{G \setminus e}(\lambda)$.

This relation gives many interesting corollaries, such as the combinatorial interpretation of $\chi_G(-1)$. It makes no sense to ask for the number of proper (-1) -colorings, but since $\chi_G(\lambda)$ is a polynomial, we can evaluate $\chi_G(-1)$. However, is this value meaningful?

Theorem 5.72. *The number of acyclic orientations, i.e. orientations with no directed cycles, of a graph G is $(-1)^{v(G)} \chi_G(-1)$.*

Proof. Let $\zeta(G)$ be the number of acyclic orientations of G . For every $e = \{u, v\} \in E$, we have

$$\zeta(G) = \zeta(G - e) + \zeta(G \setminus e).$$

This is because in an acyclic orientation of $G - e$, either there is no directed path from u to v , or there is no directed path from v to u . So e can be oriented at least in a direction. In addition, e can be oriented in two directions iff identifying u and v produces an acyclic orientation. $\square$