6
Systems of Distinct Representatives

Let's play a game as follows. You give me five cards at random from a standard 52-card deck. I keep one and show the other four cards (in a particular order) to the teaching assistant. The TA looks at these cards and can successfully announce the name of the fifth card. Do you believe it is magic or a mathematical trick?

6.1 Graph matchings and Hall's theorem

Definition 6.1 (System of distinct representatives, transversal)**.** A *system of distinct representatives* (SDR, a.k.a. *transversal*) of a family of sets $S_1, S_2, \dots, S_n$ is a sequence of n *distinct* elements $a_1, a_2, \dots, a_n$ such that $a_i \in S_i$ for all $1 \le i \le n$.

Clearly, a necessary condition for the existence of SDR is $\forall\, T \subseteq [n]$,

$$\left| \bigcup_{i \in T} S_i \right| \ge |T| \; .$$

Surprisingly, such a simple condition is also sufficient. We now reformulate this condition, and introduce *Hall's marriage theorem*, in terms of graph theory.

Definition 6.2 (Matching)**.** A *matching* in G is a subset M of edges such that no vertex is incident with more than one edge in M.
The *maximum matching* is the matching with maximum cardinality.
A *perfect matching* is a matching M such that every vertex is incident to exactly one edge in M.
A *perfect matching* for a set $U \subseteq V$ if every vertex in U is matched.

In other words, a *matching* is an *independent set* of edges. A graph on n vertices has a perfect matching if and only if its maximum matching has size $n/2$.

Theorem 6.3 (Hall's marriage theorem)**.** *In a bipartite graph (U, V, E), there is a perfect matching for U iff for any subset $W \subseteq U$, $|N(W)| \geq |W|$.*

Equivalently, Hall's theorem states that a set family has a SDR iff $\forall\, T \subseteq [n]$, $|\bigcup_{i \in T} S_i| \geq |T|$.

Proof. The necessity is clear. We only need to show sufficiency. We prove it by induction on $|U|$.

The base case $|U| = 1$ is trivial. Now suppose $|U| = n$ ($n \geq 2$) and the theorem holds for any $|U| < n$.

- **Case 1:** for every nonempty $W \subsetneq U$, $|N(W)| > |W|$. Now pick any $u \in U$. By Hall's condition, $|N(u)| \geq 1$. Match u to some $v \in N(u)$. Then consider the graph G' induced by $U \setminus \{u\}$ and $V \setminus \{v\}$. For every $W \subseteq U \setminus \{u\}$, $|N(W) \setminus \{v\}| \geq |N(W)| - 1 \geq |W|$. By induction, there is a perfect matching in G' for $U \setminus \{u\}$.
- **Case 2:** there exists a nonempty set $W \subsetneq U$ such that $|N(W)| = |W|$. Suppose $|W| = k$. Since $|W| < n$, by induction, there is a perfect matching for W. Now consider the graph G' induced by $U \setminus W$ and $V \setminus N(W)$. For any subset $W' \subseteq U \setminus W$, we have

$$\left|N(W' \cup W)\right| \geq \left|W' \cup W\right| = \left|W'\right| + k$$

by Hall's condition. Note that $|N(W)| = k$. So it follows that

$$\begin{aligned} \left|N(W') \cap (V \setminus N(W))\right| &= \left|N(W' \cup W) \cap (V \setminus N(W))\right| \\ &= \left|N(W' \cup W) \setminus N(W)\right| \\ &\geq \left|N(W' \cup W)\right| - k \quad \geq \left|W'\right| , \end{aligned}$$

which is the Hall's condition in G'. Since $|U \setminus W| < n$, by induction, there is a perfect matching in G' for $U \setminus W$. ☐

Example 6.4. Any k-regular bipartite graph has a perfect matching.

Corollary 6.5 (Birkoff's theorem)**.** *The edge set of a k-regular bipartite graph can be partitioned into k perfect matchings.*

Now we consider the problem of maximum matchings.

Definition 6.6. In a bipartite graph $G = (U, V, E)$, for any $W \subseteq U$, the *deficiency* of W is defined to be $\max\{0, |W| - |N(W)|\}$.

Hall's theorem says that if all $W \subseteq U$ has deficiency 0, then there is a perfect matching for U. The following is an easy corollary (generalization) of Hall's theorem.

Corollary 6.7. *In a bipartite graph (U, V, E), the size of the maximum matching is $|U| - D_U$ (equivalently, $|V| - D_V$), where D_U (resp. D_V) is the maximum deficiency over all subsets of U (resp. V).*

Proof. Clearly the size of maximum matching cannot be greater than $|U| - D_U$, since there exists a set $W \subseteq U$ only has $|W| - D_U$ neighbors. To see there is a matching of size $|U| - D_U$, add D_U vertices to V, where each of them is adjacent to all vertices in U. The new graph has a perfect matching by Hall's theorem. □

6.2 Vertex covers and König's theorem

The problem of maximum matchings is the *dual* of the problem of minimum vertex covers.

Definition 6.8 (Vertex cover)**.** A *vertex cover* in G is a subset C of vertices such that each edge is incident to at least one vertex in C. The *minimum vertex cover* is the vertex cover with minimum cardinality.

It is easy to see that the size of minimum vertex cover is not less than the size of maximum matching, since a vertex can only cover at most one edge in the matching. The following König's theorem, also known as König-Egerváry theorem, reveals the relation between the maximum matching and the minimum vertex cover in bipartite graphs.

Theorem 6.9 (König's theorem)**.** *In a bipartite graph, the size of the maximum matching equals the size of the minimum vertex cover.*

Remark 6.10. It is not true in general graphs. See, e.g., C_5.

We first give a proof of König's theorem by Hall's theorem.

Proof. Let $G = (U, V, E)$ be a bipartite graph. For any $W \subseteq U$, $(U \setminus W) \cup N(W)$ is a vertex cover. Combining with Corollary 6.7 and the fact that the size of a matching is not greater than the size of a vertex cover, we complete the proof. □

König's theorem is actually a special case of the *max-flow-min-cut theorem*, and also a reformulation of *strong duality*. To see what does the "dual problem" mean, we introduce a proof based on the duality of linear programming.

Proof. Proof by duality of linear programming We can write an integer programming formulation of the maximum matching problem. Any matching $M \subseteq E$ can be represented by $|E|$ variables such that $x_e = 1$ if the edge $e \in M$ and $x_e = 0$ otherwise. Conversely, any assignment to $\{x_e\}_{e \in E}$ can represent a matching if $x_e \in \{0,1\}$ and $\sum_{(u,v) \in E} x_{(u,v)} \leq 1$ for all $v \in V$. So the problem can be formulated as

$$\begin{aligned} \max \quad & \sum_{e \in E} x_e \\ \text{subject to} \quad & \forall\, e \in E,\ x_e \in \{0,1\}\,; \\ & \forall\, v \in V,\ \sum_{(u,v) \in E} x_{(u,v)} \leq 1\,. \end{aligned}$$

If we relax the constraints $x_e \in \{0,1\}$ to be $x_e \geq 0$, it can be reformulated as a linear program

$$\begin{aligned} \max \quad & \sum_{e \in E} x_e \\ \text{subject to} \quad & \forall\, e \in E,\ x_e \geq 0\,; \\ & \forall\, v \in V,\ \sum_{(u,v) \in E} x_{(u,v)} \leq 1\,. \end{aligned}$$

The relaxed problem is called the *maximum fractional matching* problem. Analogously, for each vertex v we can assign a variable y_v to represent whether v is in the vertex cover. We relax the constraints $y_v \in \{0,1\}$ again. Then we obtain the problem of *maximum fractional vertex cover* as follows:

$$\begin{aligned} \min \quad & \sum_{v \in V} y_v \\ \text{subject to} \quad & \forall\, v \in V,\ y_v \geq 0\,; \\ & \forall\, (u,v) \in E,\ y_u + y_v \geq 1\,. \end{aligned}$$

It is easy to verify that the fractional minimum vertex cover problem is the dual of the fractional maximum matching problem. So by the duality of linear programming, in any graph the size of the maximum fractional matching equals the size of the minimum fractional vertex cover.

In a bipartite graph G, we now show that the size of the maximum fractional matching equals the size of the maximum matching. Given a fractional matching $\{x_e\}_{e \in E}$, consider the subgraph consisting of fractional edges e where $x_e \notin \{0,1\}$.

- **Case 1:** There exists a cycle $\{v_1, v_2, \ldots, v_\ell\}$. Note that ℓ is an even number since G is bipartite. Let $\varepsilon = \min\{1 - x_{(v_1,v_2)}, x_{(v_2,v_3)}, 1 - x_{(v_3,v_4)}, x_{(v_4,v_5)}, \ldots, 1 - x_{(v_{\ell-1},v_\ell)}, x_{(v_\ell,v_1)}\}$. Then add ε to $x_{(v_i,v_{i+1})}$ for all odd i, and subtract ε from $x_{(v_i,v_{i+1})}$ for all even i (we assume that $v_{\ell+1} = v_1$). The resulting $\{x_e\}$ satisfy all constraints and the size of the fractional matching remains the same.

- **Case 2:** There is no cycles. Then choose any path $\{v_1, \ldots, v_\ell\}$. Note that all edges e incident to v_1 has $x_e \in \{0,1\}$ except $x_{(v_1,v_2)}$. So $x_e = 0$ if v_1 belongs to e but $e \neq (v_1, v_2)$. Similarly, $x_e = 0$ if v_ℓ belongs to e but $e \neq (v_{\ell-1}, v_\ell)$. Again, let $\varepsilon = \min\{1 - x_{(v_1,v_2)}, x_{(v_2,v_3)}, 1 - x_{(v_3,v_4)}, x_{(v_4,v_5)}, \ldots, x_{(v_{\ell-1},v_\ell)}$ or $1 - x_{(v_{\ell-1},v_\ell)}\}$. Then subtract ε from $x_{(v_i,v_{i+1})}$ for all odd i, and add ε to $x_{(v_i,v_{i+1})}$ for all even i (we assume that $v_{\ell+1} = v_1$). Now the resulting $\{x_e\}$ satisfy all constraints and the size of the fractional matching is nondecreasing.

Each operation decrease the number of fractional edges. So there is no fractional edges after finite many operations. Consequently, any fractional matching can be converted into an integral matching that is not worse, which implies that the size of the maximum fractional matching equals the size of the maximum matching.

We can also show that the size of the minimum fractional vertex cover equals the size of the minimum vertex cover in a bipartite graph $G = (U, V, E)$. We construct a vertex cover C as follows. Pick a real number $p \in [0,1]$ uniformly at random. For every $u \in U$, let $u \in C$ if $0 \le p \le y_u$, and for every $v \in V$, let $v \in C$ if $1 - y_v \le p \le 1$. Now it is easy to see that for every $u \in U$ and $v \in V$, if $(u, v) \in E$, then $y_u + y_v \ge 1$. So at least one of $\{u, v\}$ is in C, which gives that C is a vertex cover. Then we calculate the expected size of C. Clearly, for any v, $\Pr(v \in C) = y_v$. Thus, by the linearity of expectation, $\mathbb{E}[|C|] = \sum_v y_v$, which is exactly the size of the minimum fractional vertex cover. Moreover, there exists $p \in [0,1]$ such that the vertex cover C' constructed by p has the size $|C'| \le \mathbb{E}[|C|]$.

Overall, we conclude that in any bipartite graph, the size of the maximum matching equals the size of the minimum vertex cover. □

Corollary 6.11 (Reformulation of König's theorem)**.** *Let A be a* 0-1 *matrix. The minimum number of lines (i.e., columns and rows) of A that can cover all* 1*'s is equal to the maximum number of* 1*'s in A, where no two* 1*'s are on the same line.*

Corollary 6.12. *In a bipartite graph G on n vertices, the size of the maximum matching is $n - \alpha(G)$.*

Proof. It is clear to see that the complement of a vertex cover is an independent set. □

Note that there is a polynomial time algorithm to compute the size of maximum matching, but finding the maximum independent set is NP-hard in general graphs. This corollary gives an algorithm for the problem of maximum independent set in bipartite graphs.

6.3 Dilworth's theorem revisit

To recognize the power of Dilworth's theorem, we show that it contains Hall's theorem as a special case. In fact, we have seen this in the proof of Sperner's theorem (cf. Proof of Theorem 4.16).

Proof of Theorem 6.3 by Dilworth's theorem. Given a bipartite graph $G = (U, V, E)$, suppose $|U| = n$, $|V| = m$, and label them $\{1, \ldots, n\}$ and $\{n+1, \ldots, n+m\}$ respectively. Construct a poset $\mathcal{P}$ on $[n+m]$ such that $i \prec j$ iff $i < j$ and $i \sim j$ in G. That is, the Hasse diagram of $\mathcal{P}$ is exactly G with orientation $U \to V$.

Exercise: Find an explicit such poset.

Now we show that the size of maximum antichain in $\mathcal{P}$ is m. Let T be an antichain, $R = T \cap U$, and $S = V \setminus T$. Note that any element in R must have its neighbors outside T, i.e., in S. Thus, $S \supseteq N(R)$, and by Hall's condition, $|S| \geq |R|$. It follows that

$$|T| = |V| - |S| + |R| \leq |V| .$$

By Theorem 4.15, m chains can cover $\mathcal{P}$. Vertices in V are in distinct chains, so there is no chain only containing a vertex in U. This gives a perfect matching for U. $\square$

Conversely, a surprising fact is that the Dilworth's theorem is a special case of König's theorem (theorefore a special case of Hall's theorem). This fact may not be so obvious, since König's theorem is about a poset does not look like a bipartite graph. But antichains and independent sets look similar in a sense, and we have already known that an independent set is the complement of a vertex cover. So we can start from this point and try to establish a connection between maximum matchings and chain covers.

Probably not surprising, since Dilworth's theorem also gives a formulation of *min-max theorems*.

Proof of Theorem 4.15 by König's theorem. Given a poset $\mathcal{P}$ on $[n]$, define a digraph G on $[n]$ where $i \to j$ iff $i \prec j$ in $\mathcal{P}$. Namely, G is the transitive-closed Hasse diagram. Then the a chain cover in $\mathcal{P}$ corresponds to a directed path partition in G.

The key observation is that, the problem of minimum path partition in a digraph can be solved by maximum matchings. Let $H = (V_1, V_2, E)$ be a bipartite graph where V_1, V_2 are two copies of $V(G)$, and then connect $v_i \in V_1$ and $v_j \in V_2$ iff $v_i \to v_j$ in G. At beginning we use n chains to partition $V(G)$, where each chain contains a unique vertex. Each edge in a matching of H combines two chains together. This relation is a bijection between path partitions of G and matchings in H. So the minimum path partition of G is $n - |M|$ where M is the maximum matching in H.

Now by König's theorem, $|M|$ is also the size of the mimimum vertex cover in H, and thus it gives an independent set I of size $2n -$

$|M|$ in H. Define an antichain T as

$$T \triangleq \{v \mid \text{Both copies of } v \text{ in } V_1 \text{ and } V_2 \text{ are in } I\} .$$

It is easy to check that if $u \neq v \in T$, then neither $u \prec v$ nor $v \prec u$. Note that $|T| \geq |I| - n = n - |M|$. So we find an antichain whose size is not less than the size of minimum chain covers, which completes the proof. □

Now we have proved that "Dilworth $\implies$ Hall", "Hall $\implies$ König" and "König $\implies$ Dilworth". So actually Dilworth's theorem, Hall's theorem and König's theorem are "equivalent".