8
Extremal Graphs with Forbidden Subgraphs

Ramsey theorem shows that if a graph is sufficiently large, any partition has a part that contains some desired structure. In this chapter, we consider a "reverse" problem: if some structure is forbidden, how large a graph can be?

8.1 From Mantel to Turán: forbidden cliques

Let's start from the simplest structure: a triangle.

Theorem 8.1 (Mantel's theorem, 1907)**.** *If a simple graph G on n vertices is triangle-free, then G has at most $\lfloor n^2/4 \rfloor$ edges.*

Remark 8.2. This bound is tight. See, e.g., G is the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

First proof. Let $G = (V, E)$ be a triangle-free graph with maximum degree Δ, and $v \in V$ has degree Δ. Then $N(v)$ is an independent set. Now we can count the number of edges as follows:

$$|E| \le \Delta + (n - 1 - \Delta)\Delta = (n - \Delta)\Delta \le \lfloor n^2/4 \rfloor . \qquad \square$$

Second proof. Since G is triangle-free, for every adjacent u and v, it follows that $d(u) + d(v) \le n$. Summing over all edges of G, we have

$$nm \ge \sum_{\{u,v\} \in E} d(u) + d(v) = \sum_{v \in V} d(v)^2 .$$

On the other hand, by Cauchy-Schwarz inequality, it yields that

$$nm \ge \sum_{v \in V} d(v)^2 \ge \frac{\left(\sum_{v \in V} d(v)\right)}{|V|} = \frac{4m^2}{n} . \qquad \square$$

Third proof. Assign each vertex v a nonnegative weight w_v such that

This method is called the *weight shifting argument*.

$\sum_{v \in V} w_v = 1$, and let

$$S \triangleq \sum_{\{u,v\} \in E} w_u w_v \,.$$

For someone who loves linear algebra, you can rewrite it as $S = \frac{1}{2}\langle \boldsymbol{w}, \boldsymbol{A}\boldsymbol{w} \rangle$, or $S = \frac{1}{2}\boldsymbol{w}^\top \boldsymbol{A}\boldsymbol{w}$, where $\boldsymbol{A}$ is the *adjacency matrix* of G. S is offen called the *Lagrangian* of G.

Note that we can let $w_v = 1/n$ for each v, and in this case it gives

$$S = \frac{m}{n^2} \,.$$

Then we would like to *shift weights* so that the weights are concentrated on a complete subgraph and not decrease the value of S. For any $u \nsim v$, let $W_u = \sum_{u' \in N(u)} W_{u'}$ and $W_v = \sum_{v' \in N(v)} W_{v'}$. So S can be rewritten as

$$S = W_u w_u + W_v w_v + \sum_{\{u',v'\} \in E, \{u,v\} \cap \{u',v'\} = \varnothing} w_{u'} w_{v'} \,.$$

Without loss of generality, assume $W_u \le W_v$. Now we can assign $w_u + w_v$ to v and 0 to u, i.e., *shifting the weight of u to v*, which does not decrease the value of S. Hence, after shifting weights finite times, there is an assignment which concentrates all of the weights on a complete subgraph of G, and gives a nondecreasing value of S. However, G is triangle-free. That is, all weights are concentrated on the endpoints of an edge, say w_x and w_y. Finally,

$$\frac{m}{n^2} \le S = \max_{w_x + w_y \le 1} w_x w_y = \frac{1}{4} \,. \qquad \square$$

In general, denote $\mathrm{ex}(n, H)$ the maximum number of edges in a n-vertex graph without H as a subgraph. So Mantel's theorem can be restated as $\mathrm{ex}(n, K_3) \le n^2/4$. A natural generalization of Mantel's theorem is to consider forbbiden cliques. The answer is given by a fundamental result of Pál Turán (Turán Pál in the native form), which initiated *extremal graph theory*.

Hungarian names are given in the "Eastern name order", with the family name followed by the given name.

Theorem 8.3 (Turán's theorem)**.** *If a graph $G = (V, E)$ on n vertices has no $(r+1)$-cliques, then*

$$|E| \le \left(1 - \frac{1}{r}\right) \frac{n^2}{2} \,.$$

Namely, $\mathrm{ex}(n, K_{r+1}) \le \left(1 - \frac{1}{r}\right) \frac{n^2}{2}$.

We first consider which kind of graphs is the extremal case? A natural guess is the *complete r-partite graphs*. Partition V into $V = V_1 \uplus V_2 \uplus \cdots \uplus V_r$ with $|V_i| = n_i$, and let $u \sim v$ iff $u \in V_i$ and $v \in V_j$ for distinct i and j. The resulting graph is a complete r-partite graph, denoted $K_{n_1, \ldots, n_r}$. It is clear that there is no K_{r+1} in such a r-partite graph. To maximize the number of edges, we hope $V_1, \ldots, V_r$ are divided fairly.

Definition 8.4 (Turán graph)**.** A *Turán graph* on n vertices and with no K_{r+1}, denoted $T(n,r)$, is a complete r-bipartite graph $K_{n_1,\dots,n_r}$, where $n_i \in \{\lfloor n/r \rfloor, \lceil n/r \rceil\}$ for all $1 \le i \le r$.

The number of edges in Turán graph $T(n,r)$ is (roughly)

$$\binom{r}{2}\frac{n^2}{r^2} = \frac{r-1}{r}\frac{n^2}{2}.$$

Turán's theorem asserts that $T(n,r)$ is an extremal graph of $\mathrm{ex}(n, K_{r+1})$. We now give some proofs of a different nature.

First proof. Prove by induction on n. If $n \le r$, it is trivial since $\mathrm{ex}(n, K_{r+1}) \le \binom{n}{2} \le \left(1 - \frac{1}{r}\right)\frac{n^2}{2}$.

Turán's original proof.

Now assume $n > r$, and $G = (V, E)$ has the maximum number of edges. Clearly G has K_r. Let A be the set of vertices of such K_r, and $B = V \setminus A$. We count the number of edges as follows:

- $e(A) = \binom{r}{2}$;
- $e(B) \le \left(1 - \frac{1}{r}\right)\frac{(n-r)^2}{2}$ by induction;
- $e(A, B) \le (r-1)(n-r)$ since each vertex in B can only be adjacent to at most $r-1$ vertices in A.

Thus, we obtain

$$\begin{aligned} e(G) &= e(A) + e(B) + e(A,B) \\ &\le \binom{r}{2} + \frac{r-1}{r}\frac{(n-r)^2}{2} + (r-1)(n-r) \\ &= \frac{r-1}{2r}\left(r^2 + (n-r)^2 + 2r(n-r)\right) \quad = \left(1 - \frac{1}{r}\right)\frac{n^2}{2}. \end{aligned}$$ □

Second proof. The second proof also uses induction. Suppose the maximum degree is Δ, and the vertex of maximum degree is v. Then $N(v)$ is a set of Δ vertices where $G[N(v)]$ does not contain a copy of K_r. Thus $e(N(v)) \le \mathrm{ex}(\Delta, K_r)$. Note that all edges other than $E(N(v))$ are incident to at least one vertex in $V \setminus N(v)$. So we have

$$e(G) \le \mathrm{ex}(\Delta, K_r) + \Delta(n - \Delta),$$

which, using induction, further gives that

$$\begin{aligned} e(G) &\le \left(1 - \frac{1}{r-1}\right)\frac{\Delta^2}{2} + n\Delta - \Delta^2 \\ &= n\Delta - \frac{r}{r-1}\frac{\Delta^2}{2} \quad \le \left(1 - \frac{1}{r}\right)\frac{n^2}{2}. \end{aligned}$$ □

Third proof. Prove by the weight shifting argument again. Define w_v, S the same as in the proof of Mantel's theorem. The same argument holds so that all weights are concentrated on a complete subgraph K_t.

Without loss of generality, suppose their weights are $w_1, \ldots, w_t$. Let $w_{t+1} = \cdots = w_r = 0$. Then we have

$$2S = 2 \sum_{1 \le i < j \le r} w_i w_j = \left(\sum_{i=1}^{r} w_i \right)^2 - \sum_{i=1}^{r} w_i^2 = 1 - \sum_{i=1}^{r} w_i^2 .$$

Applying the Cauchy-Schwarz inequality, which gives

$$\sum_{i=1}^{r} w_i^2 \ge \frac{1}{r} \left(\sum_{i=1}^{r} w_i \right)^2 = \frac{1}{r} ,$$

we conclude that

$$\frac{2m}{n^2} \le 2S \le 1 - \frac{1}{r} . \qquad \square$$

Fourth proof. Let $G = (V, E)$ be a graph on n vertices without a K_{r+1} clique, and with the maximum number of edges. We need the following claim.

Claim 8.5. G does not contain three vertices u, v, w such that $u \nsim w$, $v \nsim w$ but $u \sim v$.

To prove this claim, for the sake of contradiction, suppose there are such vertices, and consider the following two cases.

- **Case 1:** $d(w) < d(u)$ or $d(w) < d(v)$. We may assume $d(w) < d(u)$. Then we *duplicate* vertex u, namely, add a new vertex u' with the same neighbor set as u, but do not add edge $\{u, u'\}$. The new graph is still K_{r+1}-free, because otherwise u' must be in the clique, and thus G itself contains a $(r+1)$-clique. Now we delete w in the new graph. Since $d(w) < d(u')$, the new graph has n vertices, no $(r+1)$-clique, but more edges than G.
- **Case 2:** $d(w) \ge d(u)$ and $d(w) \ge d(v)$. Then we duplicate vertex w twice, and delete u, v. Again, it is easy to see that the new graph is still K_{r+1}-free. But the new graph has more edges than G, because two copies of w contributes $2d(w)$ edges, and u, v contributes $d(u) + d(v) - 1$ edges.

Now applying Claim 8.5, it is easy to see that for any v, $V \setminus N(v)$ (i.e., $\{v\} \cup \{u \mid u \nsim v\}$) forms an independent set, and all vertices in it are adjacent to all other vertices not in the set. Thus, G has to be a complete k-partite graph for some $k \le r$. As we have already known, $T(n, r)$ gives the most edges among all complete r-partite graphs (allowing empty parts). $\square$

Corollary 8.6. *Turán graphs are the only possible extremal graphs.*

Now we give a general upper/lower bound of $\mathrm{ex}(n, H)$ for an arbitrary H. It depends on the chromatic number of H. Clearly, Turán graph $T(n,r)$ is r-colorable. So if $\chi(H) \geq r+1$, H cannot be a subgraph of $T(n,r)$. Thus we have $\mathrm{ex}(n,H) \geq (1-\frac{1}{r})\frac{n^2}{2}$. You may guess that we probably can do better. But in fact, the following result shows that this bound is asymptotically optimal (quite surprising).

Theorem 8.7 (Erdős-Stone-Simonovits theorem)**.**

$$\mathrm{ex}(n,H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right)\binom{n}{2}.$$

This result is reputed to be the *fundamental theorem in extremal graph theory* (first by Béla Bollobás).

8.2 Forbidden even cycles

The Erdős-Stone-Simonovits resolves the question of (asymptotically) determining $\mathrm{ex}(n,H)$ for many graphs H, but says nothing about bipartite graphs. Indeed, if $\chi(H) = 2$ then the result is that $\mathrm{ex}(n,H) = o(n^2)$, which is very imprecise. What is the correct asymptotic order of $\mathrm{ex}(n,H)$?

Let's start from C_4.

Theorem 8.8.

$$\mathrm{ex}(n,C_4) \leq \frac{n}{4}\left(1+\sqrt{4n-3}\right).$$

Proof. Consider the number of triples (u,v,w) such that $u \sim w$ and $v \sim w$. For each pair (u,v), there exists at most one w such that $u \sim w$ and $v \sim w$. For any vertex w, the number of its corresponding triples is at most $\binom{d(w)}{2}$. Therefore, we have

$$\begin{aligned}
&\sum_u \binom{d(u)}{2} \leq \binom{n}{2} \\
\Longrightarrow\ &\sum_u d(u)^2 \leq n(n-1) + \sum_u d(u) \\
\Longrightarrow\ &\frac{(2|E|)^2}{n} \leq n(n-1) + 2|E| \qquad\qquad \text{(Cauchy–Schwarz)} \\
\Longrightarrow\ &|E| \leq \frac{n}{4}(\sqrt{4n-3}+1).
\end{aligned}$$

□

This bound is almost best. Here we give an extremal example, which asymptotically achieves this bound.

Example 8.9. Let p be a sufficiently large prime, and $n = p^2 - 1$. Now we define $G = (V,E)$ as

$$\begin{aligned}
V &= \mathbb{F}_p^2 \setminus \{(0,0)\}, \\
E &= \{\{(a,b),(x,y)\} \mid ax+by=1\}.
\end{aligned}$$

It is easy to check that G is C_4-free, since

$$\begin{cases} a_1 x + b_1 y = 1 \\ a_2 x + b_2 y = 1 \end{cases}$$

has at most one solution for distinct (a_1, b_1) and (a_2, b_2). Moreover, the degree of each vertex (x, y) is p, or $p-1$ (if $x^2 + y^2 = 1$). Thus G has $p^2 - 1$ vertices and $\left(\frac{1}{2} + o(1)\right) p^3$ edges.

A more sharp example requires more linear algebra. We will talk about it later.

Theorem 8.10 (Bondy & Simonovits, 1974). *Let $t \geq 2$. There exists $c > 0$ such that* $\operatorname{ex}(n, C_{2t}) \leq cn^{1+1/t}$.

It is still an open problem if Bondy-Simonovits is the correct asymptotic. We know that it is asymptotically optimal if $t = 2, 3, 5$, but not for any other values of t.

8.3 Zarankiewicz's problem

We now consider the extremal problem of forbidden complete bipartite graphs.

Theorem 8.11. *Let $t \geq 2$. There exists a constant c such that* $\operatorname{ex}(n, K_{t,t}) \leq cn^{2-1/t}$.

Proof. Suppose $m = |E|$. Consider the number of t-claws, that is, the number of $(u, \{v_1, \ldots, v_t\})$ where $u \sim v_i$ for $1 \leq i \leq t$. For any t vertices $\{v_1, \ldots, v_t\}$, there exists at most $(t-1)$ such u. For any vertex u, the number of its corresponding t-claws is $\binom{d(u)}{t}$. Therefore, we have

$$\sum_u \binom{d(u)}{t} \leq (t-1)\binom{n}{t}.$$

Notice that the function $x \mapsto \binom{x}{t}$ is convex. By Jensen's inequality,

$$\sum_u \binom{d(u)}{t} \geq n \cdot \binom{2m/n}{t}.$$

Since $\frac{1}{t!}(x-t)^t \leq \binom{x}{t} \leq \frac{1}{t!}x^t$, we have

$$\frac{n}{t!}(2m/n - t)^t \leq n \cdot \binom{2m/n}{t} \leq (t-1)\binom{n}{t} \leq (t-1)\frac{n^t}{t!}.$$

It follows that

$$2m/n - t \leq (t-1)^{1/t} n^{1-1/t}$$

and further implies

$$m \leq \frac{1}{2}(t-1)^{1/t} n^{2-1/t} + \frac{1}{2} tn \,,$$

which completes the proof. □

The same argument can be applied to the extremal problems in bipartite graphs, which is equivalent to the following *Zarankiewicz's problem*.

Question 8.12 (Zarankiewicz's problem)**.** How many 1's can an $n \times n$ 0-1 matrix contain if it has no $a \times b$ submatrix whose entries are all 1's?

Denote $k_{a,b}(n)$ the maximum number of 1's in a $n \times n$ matrix with no all 1 submatrix. Usually we omit a subscript if $a = b$.

Theorem 8.13 (Kővári-Sós-Turán)**.** *Assuming* $2 \leq a \leq b$, *we have*

$$k_{a,b}(n) \leq O(b^{1/a} n^{2-1/a}) \,.$$

Remark 8.14. In fact, we can show $\mathrm{ex}(n, K_{a,b}) \leq k_{a,b}(n) \leq \mathrm{ex}(2n, K_{a,b})$. Note that $\mathrm{ex}(n, K_{a,b}) = o(n^2)$. So $k_{a,b}(n)$ and $\mathrm{ex}(n, K_{a,b})$ only differ at a constant factor.

A well-known conjecture is $k_{a,b}(n) = \Theta(n^{2-1/a})$, which is widely open. For general $n \geq t \geq 2$, a probabilistic argument can give a lower bound

$$k_t(n) \geq O(n^{2-2/t}) \,.$$

In particular, an algebraic construction similar to Example 8.9 gives that

$$k_2(n) \geq (1 + o(1)) n^{3/2} \,,$$

which matches the upper bound Theorem 8.13.

Zarankiewicz's problem is also related to the following problem in *incidence geometry*: on the $\mathbb{R}^2$ plane there are n points and n lines, how many "point-belongs-to-line" incidences between them?

Let P denote the set of points, L denote the set of lines, and $I(P, L)$ denote the set of incidence

$$I(P, L) \triangleq \{(p, \ell) \in P \times L \mid p \in \ell\} \,.$$

Since two points determine a unique line, there is no $K_{2,2}$ in the incidence matrix of P and L. Thus the result of $k_2(n)$ immediately gives that $|I(P, L)| \leq O(n^{3/2})$. Unfortunately it is not the correct answer.

The Szemerédi–Trotter theorem (Theorem 9.3) gives an asymptotically tight bound, which shows that if $|P| = |L| = n$, then

$$\max |I(P, L)| = \Theta(n^{4/3}).$$

The reason is that no all 0-1 matrices could be an incidence matrix of points and lines on $\mathbb{R}^2$ plane. For example, it is impossible to put 7 points and 7 lines on the plane such that every point lies on 3 distinct lines and every line contains exactly 3 points. However, this incidence can be realized on *projective planes*. It is actually the *Fano plane*, the smallest possible projective plane.

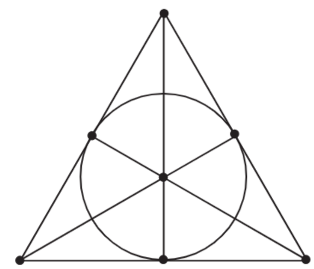

Figure 8.1: Fano plane, consisting of 7 points and 7 lines

An interesting fact is that we often make use of incidence matrices in some spaces to design extremal graphs. For instance, Example 8.9 can be viewed as incidences between points $\{(x, y) \in \mathbb{F}_p^2 \mid (x, y) \neq (0, 0)\}$ and lines $\{\ell_{a,b} \mid \ell_{a,b} : ax + by = 1 \text{ in } \mathbb{F}_p^2\}$.