9
Discrete Geometry and Extremal Set Systems

9.1 Extremal systems on points and lines

Why can't we embed the Fano plane into the $\mathbb{R}^2$ plane?

Theorem 9.1 (Sylvester–Gallai theorem)**.** *For every n points in the plane, if they are all on the same line, then there is a line that contains exactly two of them.*

Proof. Let $(p_0, \ell_0) \in P \times L$ be the pair such that $p_0 \notin \ell_0$ and the distance between p_0 and ℓ_0 are smallest among all pairs. Then ℓ_0 is a desired line. $\square$

The Sylvester–Gallai theorem therefore shows that the Fano plane cannot be embedded into the real plane, because each point p lies on three lines where each line contains two points other than p.

We may ask how many such two-point lines every n-point configuration in the plane must contain. In 2012, Ben Green and Terence Tao showed that for any sufficiently large n, if n is even then there are at least $n/2$ two-point lines, and if n is odd, there are even at least $3\lfloor n/4 \rfloor$ such lines. Their result is the best possible.

The Sylvester–Gallai theorem directly implies another famous result on points and lines in the plane, due to Paul Erdős and Nicolaas G. de Bruijn.

Theorem 9.2 (de Bruijn-Erdős theorem)**.** *Every noncollinear set of n points in the plane determines at least n distinct lines.*

Proof. Prove by induction on n. The base case is $n = 3$, which is trivial. Suppose $n > 3$ and that the statement is true for $n - 1$. By Theorem 9.1, there is a line containing exactly two points, say, p_0 and p_1. Let $p_2, \ldots, p_{n-1}$ be other $n - 2$ points. If $p_1, p_2, \ldots, p_{n-1}$ are collinear, then p_0 does not lie on this line, and thus $p_0p_1, p_0p_2, \ldots, p_0p_{n-1}$

determines other $n-1$ distinct lines. Otherwise, by induction, $p_1, \ldots, p_{n-1}$ determines at least $n-1$ distinct lines, where each of them differs from line $p_0 p_1$. $\square$

Another related question is how many incidence relations n points and n lines can determine. As we mentioned before, let P denote the set of points, L denote the set of lines, and $I(P, L)$ denote the set of incidence

$$I(P,L) \triangleq \{(p,\ell) \in P \times L \mid p \in \ell\}.$$

This is a fundamental problem in the area called *incidence geometry*, where we only consider points, lines and their incidence relation.

Then the following Szemerédi–Trotter theorem gives an asymptotically optimal bound.

Theorem 9.3 (Szemerédi–Trotter theorem)**.** $|I(P,L)| \lesssim |P|^{2/3}|L|^{2/3} + |P| + |L|$.

Example 9.4. Let $P = \{(x,y) \mid x \in [k], y \in [2k^2]\}$ and $L = \{y = ax + b \mid a \in [k], b \in [k^2]\}$. Each line contains exactly k points, which implies that $|I(P,L)| = O(k^4)$ while $|P| = 2|L| = 2k^3$.

A useful observation is that there is no 2×2 all 1 submatrix in the incidence matrix of points and lines by *Euclid's postulates*.

Assumption 9.5 (Euclid's postulates)**.** Two points determine a unique line. Two lines intersect at a unique point (if they are not parallel).

We reformulate the proof of Theorem 8.8 to give an "easy bound" to the incidence problem:

$$\begin{aligned}
|I(P,L)|^2 &= \left(\sum_{\ell \in L} \sum_{p \in P} [p \in \ell]\right)^2 \\
&\le |L| \cdot \sum_{\ell \in L} \left(\sum_{p \in P} [p \in \ell]\right)^2 \qquad \text{(Cauchy-Schwarz inequality)} \\
&= |L| \cdot \sum_{p_1, p_2 \in P} \sum_{\ell \in L} [p_1 \in \ell] \cdot [p_2 \in \ell] \\
&= |L| \cdot \left(\sum_{p \in P} \sum_{\ell \in L} [p \in \ell] + \sum_{p_1 \neq p_2} \sum_{\ell \in L} [p_1 \in \ell \wedge p_2 \in \ell]\right) \\
&\le |L| \cdot (|I(P,L)| + |P|^2).
\end{aligned}$$

It implies that $|I(P,L)| \lesssim |P| \cdot |L|^{1/2} + |L|$. If $|P| \approx |L| = O(n)$, we have $|I(P,L)| \le O(n^{3/2})$.

For convienience, we write $f \lesssim g$ to denote $f \le O(g)$ and write $f \approx g$ to denote $f = O(g)$.

However, only applying Assumption 9.5 is not sufficient to obtain the asymptotically tight bound. To get a better result, we roughly sketch the idea here. Suppose we can evenly divide the plane into D^2 cells such that each cell contains $|P|/D^2$ points and $|L|/D$ lines, and each line enters D cells. Then we apply the easy bound to each cell to get

$$|I_{\text{cell}}(P,L)| \lesssim \frac{|P|}{D^2}\left(\frac{|L|}{D}\right)^{1/2} + \frac{|L|}{D},$$

and thus

$$|I(P,L)| \approx D^2 |I_{\text{cell}}(P,L)| \lesssim |P| \left(\frac{|L|}{D}\right)^{1/2} + D|L| .$$

Choosing $D = |P|^{2/3} |L|^{-1/3}$, it follows that $|I(P,L)| \lesssim |P|^{2/3} |L|^{2/3}$. In fact, this idea can be realized by both the *probabilistic method* (Section 10.3) and the *polynomial method*. We will introduce details later.

Probably a surprising fact is that de Bruijn–Erdős theorem (i.e., Theorem 9.2) can be derived by only Eculid's postulate (i.e., Assumption 9.5).

Theorem 9.6. *Let U be a set of $n \geq 3$ elements, and $S_1, \ldots, S_m$ be proper subsets of U, such that every pair of elements of U is contained in precisely one set S_i. Then it holds that $m \geq n$.*

Proof. Assume each S_i has size at least 2. Otherwise remove all singleton sets, which does not change the condition, but decrease the value of m.

For any $u \in U$, let r_u be the number of sets S_i containing u. Note that each S_i is a proper subset, so $r_u \geq 2$. For any u and S_i such that $u \in S_i$, there exists $v \neq u \in S_i$ and $w \notin S_i$. Thus there exists a set S_j such that $v, w \in S_j$ but $u \notin S_j$. This gives that $r_u < m$. Moreover, if $u \notin S_i$, then $r_u \geq |S_i|$, since the sets containing u and an element in S_i must be distinct. Now, suppose $m < n$, then we have $mn - m|S_i| > mn - nr_u$ for $u \notin S_i$, and thus

$$1 = \sum_{u\in U} \frac{1}{n} = \sum_{u\in U} \sum_{u\notin S_i} \frac{1}{n(m-r_u)} > \sum_{S_i} \sum_{u\notin S_i} \frac{1}{m(n-|S_i|)} = \sum_{S_i} \frac{1}{m} = 1,$$

which leads to a contradiction. □

Xiaomin Chen and Vesěk Chvátal have even partially extended the de Bruijn-Erdős theorem to general *metric spaces*.

9.2 Intersecting set families

We now consider the intersection relation between sets. A family of sets is called *intersecting* if any two of them have a nonempty intersection. Let U be a set of n elements. We can ask the following questions:

- how many subsets of U are there at most such that each pair of subsets is *intersecting*?
- how many nonempty subsets of U are there at most such that none pair of subsets is *intersecting*?

However both of them are less interesting, because the answer to the second one is trivially $n+1$. For the first one, by the pigeonhole principle, it is clear that the number of intersecting subsets cannot exceed 2^{n-1}, and this bound can be easily achieved by picking all subsets containing 1.

An interesting problem is to find the largest intersecting family of k-element sets. Clearly, if $2k > n$, then $\binom{U}{k}$ is an intersecting family. So we only consider $2k \leq n$. The lower bound is easy. All k-element sets containing 1 is an intersecting family of size $\binom{n-1}{k-1}$. But can we do better?

Theorem 9.7 (Erdős–Ko–Rado theorem)**.** *The largest size of an intersecting family of k-element subsets of* $[n]$ *is* $\binom{n-1}{k-1}$ *when* $n \geq 2k$.

Paul Erdős, Chao Ko and Richard Rado found this result in 1938, but it was not published until 23 years later.

Fisrt proof. The basic idea is to find a special type of extremal intersecting families, so that we can prove the theorem by induction. We define a *shift* operator as follows. Given $\mathcal{F} \subseteq 2^{[n]}$ and $F \in \mathcal{F}$, the (i,j)-*shift* is

Erdős' *shifting technique*.

$$S_{i,j}(F) \begin{cases} (F \setminus \{j\}) \cup \{i\} & \text{if } i \notin F,\ j \in F, \text{ and } (F \setminus \{j\}) \cup \{i\} \notin \mathcal{F}\,, \\ F & \text{otherwise}\,. \end{cases}$$

and the (i,j)-*shift* for family $\mathcal{F}$ is

$$\mathcal{S}_{i,j}(\mathcal{F}) = \{S_{i,j}(F) \mid F \in \mathcal{F}\}\,.$$

Roughly speaking, the (i,j)-shift operator is to replace j with i for each set in $\mathcal{F}$, if possible. By definition, we have $|S_{i,j}(F)| = |F|$ and $|\mathcal{S}_{i,j}(\mathcal{F})| = |\mathcal{F}|$. A key observation is the following claim.

Claim 9.8. If $\mathcal{F}$ is intersecting, so is $\mathcal{S}_{i,j}(\mathcal{F})$.

Otherwise, suppose $S_{i,j}(F) \cap S_{i,j}(G) = \emptyset$. Clearly, it cannot happen when both F, G are shifted, or neither F, G is shifted. Without loss of generality, we may assume $S_{i,j}(F) = F$ and $S_{i,j}(G) \neq G$. So $j \in G$ and $i \notin G$. Since $F \cap G \neq \emptyset$, and $F \cap S_{i,j}(G) = \emptyset$, we know that $F \cap G = \{j\}$, and $i \notin F$. When performing the (i,j)-shift on F, why did not we replace j with i? The only reason is $F' = (F \setminus \{j\}) \cup \{i\} \in \mathcal{F}$. But $F' \cap G \neq \emptyset$, which contradicts to $F \cap G = \{j\}$.

Now we can give a proof by induction on n and k. The base case is $n = 2$. So the only possible value for k is 1, which is trivial. Next, suppose the statement is true for $n-1$. We argue that it is also true for n. If $k = 1$, it is trivial. If $n = 2k$, note that

$$\binom{n-1}{k-1} = \frac{1}{2}\binom{n}{k}.$$

By the pigeonhole principle, any subset family of size $> \binom{n-1}{k-1}$ contains some k-element set F and its complement at the same time, which is not intersecting.

For the case $n > 2k$, we shift the intersecting family $\mathcal{F}$, namely, apply $\mathcal{S}_{i,j}$ to $\mathcal{F}$ for every $i \in [n-1]$ and $j = n$. Now let

$$\mathcal{F}_n = \{F \in \mathcal{F} \mid n \in F\} \quad \text{and} \quad \mathcal{F}_{\overline{n}} = \mathcal{F} \setminus \mathcal{F}_n .$$

For $\mathcal{F}_{\overline{n}}$, it is an intersecting family of subsets on $[n-1]$, and thus, by induction, it has size $|\mathcal{F}_{\overline{n}}| \le \binom{n-2}{k-1}$. For $\mathcal{F}_n$, let $F, G \in \mathcal{F}_n$ be two subsets, and $F' = F \setminus \{n\}, G' = G \setminus \{n\}$ respectively. Since $|F'| + |G'| = 2k-2 < n-1$, there exists $t \in [n-1]$ such that $t \notin F \cup G$. Again, why did not we replace n with t in F and G when applying $\mathcal{S}_{t,n}$ to $\mathcal{F}$? Of course both $F' \cup \{t\}$ and $G' \cup \{t\}$ are in $\mathcal{F}$. But $(F' \cup \{t\}) \cap G \neq \emptyset$. So we obtain $F' \cap G' \neq \emptyset$, which implies that $\mathcal{F}_n$ is also an intersecting family if we remove n from each element in $\mathcal{F}_n$. Thus $\mathcal{F}_n$ has size at most $\binom{n-2}{k-2}$ by induction. We conclude the theorem by

Here, we also need to note that any set in $\mathcal{F}$ not containing n does not change during shifting.

$$\binom{n-1}{k-1} = \binom{n-2}{k-1} + \binom{n-2}{k-2}. \qquad \square$$

Second proof. The following great idea, due to G.O.H. Katona, is a double counting by connecting intersecting sets and consecutive subsequences in circular permutations.

Katona's circle.

Let π be a circular permutation on $[n]$, and C_π be the set of all consecutive subsequences of length k in π. Namely, C_π is given by

$$C_\pi = \{(\pi_{s+1}, \pi_{s+2}, \ldots, \pi_{s+k}) \mid s = 0, 1, 2, \ldots, n-1\},$$

where $\pi(n+t) = \pi(t)$ for $1 \le t \le n$. Let $\mathcal{F}$ be an intersecting family. A key observation is the following.

Claim 9.9. $|\mathcal{F} \cap C_\pi| \le k$.

Suppose $\mathcal{F} \cap C_\pi \neq \emptyset$. Let $B_0 = (\pi_{s+1}, \pi_{s+2}, \ldots, \pi_{s+k}) \in \mathcal{F} \cap C_\pi$, and $B_t = (\pi_{s+t+1}, \pi_{s+t+2}, \ldots, \pi_{s+t+k})$ for $1 \le |t| \le k-1$. Clearly in C_π only above $2(k-1)$ subsequences $B_{(k-1)}, \ldots, B_{-1}, B_1, \ldots, B_{k-1}$ are intersecting with B_0. Since for each t, $B_t \cap B_{t+k} = \emptyset$. Thus these $2(k-1)$ subsequences can be partitioned into $k-1$ pairs such that

$\mathcal{F}$ contains at most one in each pair. By the pigeonhole principle, we conclude Claim 9.9.

We now count in two ways the sum L of $|\mathcal{F} \cap C_\pi|$ over all circular permutations. By the claim, $L \le k(n-1)!$. But for each $F \in \mathcal{F}$, there are $k!(n-k)!$ circular permutations that contains F as a consecutive subsequence. It implies that $L = |\mathcal{F}|\, k!(n-k)!$, and hence

$$|\mathcal{F}| \le \frac{k(n-1)!}{k!(n-k)!} = \binom{n-1}{k-1}. \qquad \square$$

Remark 9.10. A natural question is whether all sets containing a same element is the only largest case. The answer if true if $n < 2k$, but it is false if $n = 2k$. Let $\mathcal{F}$ be a set family such that for every size-k subset $S \in \binom{U}{k}$, exactly one of S and $\overline{S}$ is contained in $\mathcal{F}$. It is easy to check that $\mathcal{F}$ is intersecting, and in most cases $\cap_{S\in\mathcal{F}} S = \emptyset$.

A related problem is to ask the size of the largest intersecting family where every pair of set has the same size intersection.

Theorem 9.11 (Fisher's inequality)**.** *Let $C_1, \dots, C_m$ be distinct subsets of $[n]$ such that $\left|C_i \cap C_j\right| = k$ for some fixed $k \in [n]$ and every $i \ne j$. Then $m \le n$.*

The whole proof requires a powerful tool: the *linear algebraic method*. We postpone the whole proof to Chapter 14. But the case $k = 1$ is equivalent to Theorem 9.6. We give a proof here.

Proof of the case $k = 1$. If there exists $j \in [n]$ such that $j \in C_i$ for all $i \in [m]$, then $C_i \setminus \{j\}$ are disjoint. Since they are subsets of an $(n-1)$-set, there are at most n such subsets. So $m \le n$.

Now assume there is no j such that $j \in C_i$. Let S_j be the set of i's such that $j \in C_i$, namely, $i \in S_j \iff j \in C_i$. So $S_j \subsetneq [m]$. Note that C_i is also the set of j's such that $i \in S_j$. By assumption, $\left|C_{i_1} \cap C_{i_2}\right| = 1$ for each pair of $i_1 \ne i_2 \in [m]$, which implies that $\{i_1, i_2\}$ is contained in precisely one set S_j. By Theorem 9.6, we conclude that $m \le n$. $\square$

On the other hand, it is easy to check that by using the same argument in reverse, Theorem 9.6 can derived from the case $k = 1$ of Fisher's inequality. So they are equivalent indeed.

9.3 *Antichains revisit*

The antichains for subset poset is an important topics for extremal set families. Recall that, an antichain for sets is a family $\mathcal{T}$ of subsets of

$[n]$ such that no set in $\mathcal{T}$ contains another set in $\mathcal{T}$. Sperner's theorem (i.e., Theorem 4.16) asserts that the largest antichain has size $\binom{n}{\lfloor n/2 \rfloor}$. We now introduce another proof, which gives a considerably sharper result.

Theorem 9.12 (LYM inequality)**.** *Let $\mathcal{T}$ be an antichain over $[n]$. Then*

$$\sum_{T \in \mathcal{T}} \binom{n}{|T|}^{-1} \leq 1\,.$$

This result is due to Lubell (1966), and was also discovered by Meshalkin (1963) and by Yamamoto (1954). The following elegant proof is also due to Lubell.

Proof. Given a permutation π on $[n]$, let P_π be the set of *prefixes* of π, defined by

$$P_\pi \triangleq \{\{\pi(1), \ldots, \pi(k)\} \mid k = 1, 2, \ldots, n\}\,.$$

For any antichain $\mathcal{T}$, note that $|P_\pi \cap \mathcal{T}| \leq 1$. So a double counting argument gives that

$$\sum_{T \in \mathcal{T}} |T|!(n - |T|)! = \sum_{\pi} |P_\pi \cap \mathcal{T}| \leq n!\,,$$

which yields that

$$1 \geq \sum_{T \in \mathcal{T}} \frac{|T|!(n-|T|)!}{n!} = \sum_{T \in \mathcal{T}} \binom{n}{|T|}^{-1}. \qquad \square$$

In fact, Lubell's result is a special case of an earlier result of Bollobás.

Theorem 9.13 (Bollobás)**.** *Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be two sequences of sets such that $A_i \cap B_j = \emptyset$ if and only if $i = j$. Then*

$$\sum_{i=1}^{m} \binom{a_i + b_i}{b_i}^{-1} \leq 1\,,$$

where $a_i = |A_i|$ and $b_i = |B_i|$ for all $1 \leq i \leq m$.

This theorem can also be proved by Lubell's method of counting permutations.

Proof. Let U be the union of all sets $A_i \cup B_i$, and assume $U = [n]$ without loss of generality. Given two disjoint subsets A and B, we say a permutation π *separates* the pair (A, B) if no element of B precedes an element of A, i.e., if $\pi(i) \in A$ and $\pi(j) \in B$ then $i < j$.

Claim 9.14. Each permutation separates at most one pair (A_i, B_i)

In fact, suppose π separates (A_i, B_i) and (A_j, B_j) for some $i \neq j$. Since $A_i \cap B_j \neq \emptyset$ and $A_j \cap B_i \neq \emptyset$, pick two elements in $A_i \cap B_j$ and $A_j \cap B_i$ respectively, and let k, ℓ be their positions in π. Namely, $\pi(k) \in A_i \cap B_j$, and $\pi(\ell) \in A_j \cap B_i$. If π separates (A_i, B_i), then $k < \ell$; if π separates (A_j, B_j), then $\ell < k$. Thus it leads to contradiction.

We now count the number of permutations separating a fixed pair. For any permutation, it separates at most one pair, so the number is at most $n!$. For each pair (A_i, B_i), the number of permutations than can separate it is

$$\binom{n}{a_i + b_i} a_i!\, b_i!\, (n - a_i - b_i)! = n! \binom{a_i + b_i}{b_i}^{-1}.$$

Summing up over all m pairs we obtain the desired inequality. □

Remark 9.15. To see that Theorem 9.12 is a special case of Theorem 9.13, let $A_1, \ldots A_m$ be all sets in an antichain, and $B_i = [n] \setminus A_i$.