11
Alteration Method

We now introduce a method called *alteration*.

11.1 Alterations

We begin with an example. Recall that (Theorem 10.1) Erdős gave a probabilistic proof to show that $R(k,k) > n$ if

$$\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1.$$

In its proof, we color the edges of K_n independently and uniformly at random. Fix a set $S \in \binom{[n]}{k}$. The probability that S induces a monochromatic K_k is $2^{1-\binom{k}{2}}$. We showed in the previous chapter that by the union bound, if $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ there exists a coloring such that no monochromatic K_k exists. However, we could do something more clever. If there exists a monochromatic K_k in the graph, we can remove an vertex from the graph to obtain a graph and a coloring without monochromatic K_k.

In fact, let X be the random variable of the number of monochromatic K_ks. Then $\mathbb{E}[X]$ is $\binom{n}{k} \cdot 2^{1-\binom{k}{2}}$. So there exists a coloring such that the number of monochromatic K_k is at most $\mathbb{E}[X]$.

For any monochromatic K_k, we delete a vertex of it from the graph. The remaining size will be at least $n - \mathbb{E}[X]$, which implies the following theorem.

Theorem 11.1. *For any k, n, we have*

$$R(k,k) > n - \binom{n}{k} \cdot 2^{1-\binom{k}{2}}.$$

Remark 11.2. Theorem 10.1 implies that

$$R(k,k) > \left(\frac{1}{e\sqrt{2}} + o(1)\right) k\, 2^{k/2}\,,,$$

while Theorem 11.1 gives a better lower bound

$$R(k,k) > \left(\frac{1}{e} + o(1)\right) k\, 2^{k/2}\,.$$

Note that for general Ramsey Number, we also have a similar lower bound.

Theorem 11.3. *For any s, t, n and $p \in [0,1]$, we have*

$$R(s,t) > n - \binom{n}{s} \cdot p^{\binom{s}{2}} - \binom{n}{t} \cdot (1-p)^{\binom{t}{2}}.$$

This method is called *alteration*. When random structures do not have all the desired properties but may have some "bad parts", we can alter the structure to remove all "bad parts".

Another classic application of alteration is the problem of *minimum dominating set*. We first give its definition as follows.

Definition 11.4 (Dominating set)**.** For any graph $G = (V, E)$, a vertex subset U is a *dominating set* if for any $v \in V$, $N^+(v) \cap U \neq \emptyset$, where $N^+(v)$ is the set of v and all its neighbors, i.e., $N^+(v) = N(v) \cup \{v\}$.

Example 11.5. A vertex cover is a dominating set. A *maximal* independent set is a dominating set.

However, for dominating sets, we would like to find the smallest one. Vertex covers and maximal independent sets may be too large. Actually, we can always find a dominating set which only depends on the minimum degrees.

Theorem 11.6. *For any graph of size n with minimum degree $\delta > 1$, there exists a dominating set of size at most $\left(\frac{\log(\delta+1)+1}{\delta+1}\right) n$.*

We first consider some naive attempts. For example, we can greedily take a vertex into the dominating set and remove all its neighbors. However, after taking the first vertex, we cannot bound the number of removed vertices at each step, since the minimum degree of vertices in the remaining graph is no longer δ.

Proof. We use the alteration method. Let $p \in [0,1]$ be a fixed parameter to be determined later. We independently pick each vertex into set X with probability p. Consider the vertices that are not dominated by vertices in X. Let $Y = V \setminus N^+(X)$. Clearly, $X \cup Y$ is a dominating set. Now, let's bound the size of $X \cup Y$.

Note that for any vertex v, $\mathbf{Pr}[v \in Y] \le (1-p)^{1+\delta}$, since neither v nor its neighbors are in X. Thus, we have

$$\begin{aligned}\mathbb{E}[|X \cup Y|] &= \mathbb{E}[|X|] + \mathbb{E}[|Y|] \\ &\le pn + (1-p)^{1+\delta} n \\ &\le (p + e^{-p(1+\delta)})n \quad \text{(minimizing by setting } p = \tfrac{\log(\delta+1)}{\delta+1}\text{)} \\ &\le \left(\frac{\log(\delta+1)+1}{\delta+1}\right) n\,,\end{aligned}$$

which completes the proof. □

Remark 11.7. We can also apply the same technique when we are searching for an independent set. We independently pick each vertex into set X with probability p. For each edge, if both of its endpoints are selected, we remove one of them. In this way, the expectation on the size of the independent set is $pn - p^2 m$. By setting $p = n/(2m)$, we can show that there exists an independent set of size at least $\frac{n^2}{4m}$, which, unfortunately, is worse than Caro–Wei inequality (cf. Theorem 10.13).

11.2 Markov's inequality

Sometimes it is required not only to remove "bad events", but also to estimate the probability of removing too many events. The following *Markov's inequality* gives a simple probabilistic tool.

Theorem 11.8 (Markov's inequality)**.** *Let X be a nonnegative random variable. Then*

$$\Pr(X \ge t) \le \frac{\mathbb{E}[X]}{t}\,.$$

Proof. $\mathbb{E}[X] = \mathbb{E}[X \mid X \ge t]\Pr(X \ge t) + \mathbb{E}[X \mid X < t]\Pr(X < t) \ge t\Pr(X \ge t)$. □

A classic example is the existence of graphs with high girth and large chromatic number. If a graph has a k-clique, then we can say $\chi \ge k$. Conversely, if χ is large, is it always possible to verify it by observing local information? Surprisingly, this is far from being true, even for "locally tree-like" graphs.

Recall that the girth of a graph is the minimum length of cycles in the graph.

Theorem 11.9 (Paul Erdős, 1959). *For any ℓ, k, there exists a graph with girth larger than ℓ and chromatic number larger than k.*

Proof. Let $G \sim \mathcal{G}(n,p)$ with $p = (\log n)^2/n$. Let X be the number of cycles of length no larger than ℓ. In K_n, there are exactly $\binom{n}{i} \cdot \frac{(i-1)!}{2}$ cycles of length i. Therefore, we have

$\mathcal{G}(n,p)$ is an Erdős-Rényi random graph on n vertices, where every two vertices are connected with probability p independently.

$$\mathbf{E}[X] = \sum_{i=3}^{\ell} \binom{n}{i} \cdot \frac{(i-1)!}{2} \cdot p^i \le \sum_{i=3}^{\ell} n^i \cdot p^i = o(n).$$

Applying Markov's inequality, we have $\mathbf{Pr}[X \ge \frac{n}{2}] = o(1) < \frac{1}{2}$.

Note that $\chi(G) \ge \frac{n}{\alpha(G)}$. It suffices to show $\alpha(G) \le \frac{n}{k}$. By setting $t = \frac{3\log n}{p}$, we have

$$\mathbf{Pr}[\alpha(G) \ge t] \le \binom{n}{t} \cdot (1-p)^{\binom{t}{2}} < n^t \cdot e^{-p \cdot \frac{(t-1)t}{2}} = o(1) < \frac{1}{2}.$$

Let n be sufficiently large such that $\mathbf{Pr}[X < \frac{n}{2} \wedge \alpha(G) < t] > 0$. For each cycle no larger than ℓ, we remove a vertex from it. Suppose the graph we get is G'. Then $|V(G')| \ge \frac{n}{2}$. Also, its girth is larger than ℓ, and $\alpha(G') \le \alpha(G) \le \frac{3\log n}{p}$, which implies that $\chi(G') \ge \frac{np}{6\log n} > k$. This completes the proof. $\square$

Remark 11.10. Constructing such a graph is not that easy. Here we introduce how to construct a triangle-free graph with large chromatic number. Let G_2 be graph K_2. Given $G_{n-1} = (V,E)$ for $n \ge 3$, construct $G_n = (V \cup V' \cup \{w\}, E \cup E')$ as follows:

- V' is a copy of V;
- For any $(u,v) \in E$, add (u',v) in E' where u' is the copy of u in V';
- For any $v' \in V'$, add (v',w) in E'.

It's easy to check that G_n is triangle-free and its chromatic number equals to n.

Actually, even if we check the chromatic number of a sub-graph induced by $\varepsilon \cdot n$ vertices, we still know nothing about the chromatic number of the whole graph. Paul Erdős proved the following theorem.

Theorem 11.11 (Paul Erdős, 1962). *For any $k > 0$, there exists $\varepsilon > 0$ such that for any sufficiently large n, there exists a graph G of size n with $\chi(G) \ge k$, while $\chi(G[S]) \le 3$ for all $|S| \le \varepsilon \cdot n$ (where $G[S]$ is the subgraph of G induced by the vertex set S).*

Proof. For a fixed k, let c, ε satisfy $c > 2\ln 2 \cdot k^2 H(1/k)$ and $\varepsilon < 3^3 e^{-5} \cdot c^{-3}$, where

$$H(x) = -x \log_2 x - (1-x)\log_2(1-x)$$

is the entropy function.

Set $p = c/n$ and let $G \sim \mathcal{G}(n, p)$. Now, let's prove that G satisfies all requirements almost surely.

Let's first discuss its chromatic number. If $\chi(G) \le k$, then $\alpha(G) \ge \frac{n}{k}$. Let X be the number of independent sets of size $\frac{n}{k}$. We have

$$\mathbf{E}[X] = \binom{n}{n/k}(1-p)^{\binom{n/k}{2}} < 2^{n(H(1/k)+o(1))} \cdot e^{-cn/2k^2 \cdot (1+o(1))},$$

which is $o(1)$ by our condition on c. This implies that $\chi(G) > k$ almost surely.

Now, let's consider the other constraint. If there exists a set of size no larger than $\varepsilon \cdot n$ such that the chromatic number of its induced sub-graph is larger than 3, let S be the minimal set such that $\chi(G[S]) = 4$. Thus, for any vertex $v \in S$, $\chi(G[S \setminus \{v\}]) = 3$ which implies that v has at least 3 neighbors in S. Therefore, there are at least $\frac{3|S|}{2}$ in $G[S]$. Let $t = |S|$ and $\mathcal{E}_t$ be the event that there exists a size-t induced sub-graph with at least $\frac{3t}{2}$ edges. We have

$$\begin{aligned}
\mathbf{Pr}[\mathcal{E}] &\le \binom{n}{t} \cdot \binom{\binom{t}{2}}{3t/2} \cdot (\frac{c}{n})^{3t/2} \\
&\le (\frac{ne}{t} \cdot (\frac{te}{3})^{3/2} (\frac{c}{n})^{3/2})^t \\
&\le (e^{5/2} \cdot 3^{-3/2} \cdot c^{3/2} \cdot \sqrt{t/n})^t.
\end{aligned}$$

Now, we can show that

$$\sum_{t \le \varepsilon \cdot n} \mathbf{Pr}[\mathcal{E}_t] \le \sum_{t \le \varepsilon \cdot n} (e^{5/2} \cdot 3^{-3/2} \cdot c^{3/2} \cdot \sqrt{t/n})^t = o(1),$$

which implies that $\forall |S| \le \varepsilon \cdot n$, $\chi(G[S]) \le 3$ holds with high probability. This completes the proof. □

11.3 Greedy random coloring

In Section 10.1, we have introduced the problem of 2-colorable hypergraphs and defined $m(k)$, which is the minimum number of edges in a k-uniform hypergraph that is not 2-colorable. We have also proved that $2^{k-1} \le m(k) \le k^2 \cdot 2^k$. Now, we will give a better bound of $m(k)$.

Theorem 11.12 (Radhakrishnan & Srinivasan, 2000). $m(k) = \Omega(2^k \cdot \sqrt{\frac{k}{\log k}})$.

Proof. (by Cherkashin & Kozik, 2015) Suppose $H = (V, E)$ with $|E| = m$. We randomly select a weight function $w : V \to [0,1]$. Also, we color all vertices according to their weights from small to large. For each vertex v, color it 0 if it does not form an all-0 edge. Otherwise, we color it 1. In this way, we construct a coloring which leads to no all-0 edge. Now, let's bound the probability that an all-1 edge exists.

We can see that for any color-1 vertex v, there exists an edge $e \in E$ such that v has the largest weight in e. This implies that for an all-1 edge f, there exists an edge e such that the vertex with the smallest weight in f has the largest weight in e. In this case, we say the edge pair (e, f) is conflicting. We try to bound the probability that a conflicting edge pair exists.

Let $L = [0, \frac{1-p}{2})$, $M = [\frac{1-p}{2}, \frac{1+p}{2}]$ and $R = (\frac{1+p}{2}, 1]$ where p is determined later. For any edge $e \in E$, $\mathbf{Pr}[\forall v \in e, w(v) \text{ lies in } L] = \mathbf{Pr}[\forall v \in e, w(v) \text{ lies in } R] = (\frac{1-p}{2})^k$. Therefore, $\mathbf{Pr}[\exists \text{ all } L \text{ or all } R \text{ edge}] \leq 2m \cdot (\frac{1-p}{2})^k$.

Now, let's assume that no all-L or all-R edge exists. If (e, f) is conflicting, then $e \cap f$ is a single vertex of which the weight value lies in M. Assume that $v = e \cap f$. Then, we have

$$\begin{aligned}\mathbf{Pr}[v \text{ lies last in } e \text{ and first in } f] &= \int_{(1-p)/2}^{(1+p)/2} x^{k-1}(1-x)^{k-1} dx \\ &\leq p/4^{k-1}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mathbf{Pr}[\exists \text{ conflicting } (e, f)] &\leq 2m\left(\frac{1-p}{2}\right)^k + m^2 \cdot \frac{p}{4^{k-1}} \\ &< 2^{1-k} m e^{-pk} + (2^{1-k} m)^2 p \\ &< 1.\end{aligned}$$

The last inequality is true when $m \leq c \cdot 2^k \sqrt{\frac{k}{\log k}}$ for some sufficiently small constant $c > 0$. This completes the proof. $\square$

Remark 11.13. What is the source of the gain in the the $L \cup M \cup R$ argument? The expected number of conflicting pairs is unchanged. It must be that we are somehow clustering the bad events by considering the event when some edge lies in L or R.