Lecture 10. Two phase Simplex method; Duality of LP

Simplex method: start from origin. move and shift.

If origin is not feasible? suppose $x=d$ is feasible.

let $y_i = d_i - x_i$. rewrite the LP so that $y=0$ is feasible.

y_i may be less than $0 \rightarrow y_i = y_i^+ - y_i^- = d_i - x_i$. origin is a vertex

How to find a feasible solution?

Consider a LP: $\min c^T x$. s.t. $Ax = b$, $x \geq 0$.

add slack variables $s_1, \ldots, s_m$. s.t. $A_i x + s_i = b_i$. $\forall i$.

a trivial solution $x_1 = x_2 = \cdots = x_n = 0$, $s_i = b_i$. $\forall i$.

consider the LP: $\min s_1 + s_2 + \cdots + s_m$. $A_i x + s_i = b_i$ $\forall i$.

if optimal $= 0$. find a feasible sol. of the initial LP. ow. infeasible.

Two more questions: correctness and running time.

Correctness: simplex method finds a local optimum.

if halts. origin is the local minimum. every coefficient is ≥ 0.

for initial LP. $x_1, \ldots, x_k$ are neighbours of x^*. $\forall y \in P$.

$y - x^*$ is a conic combination of $(x_1 - x^*)$, $(x_2 - x^*)$, $\ldots$, $(x_k - x^*)$.

Running time: avoid singular bad instance by perturbation. (smooth analysis).

Consider a LP $\min\ 3x_1+2x_2$ s.t. $x_1+x_2\le 5,\ x_1\le 3.\ x_2\le 4.$

Trivially $x_1=x_2=0$ is the optimal solution since $x_1, x_2\ge 0$

How about $\min\ -3x_1-2x_2$? (equivalent to $\max\ 3x_1+2x_2$).

$3x_1+2x_2\le 13$ since $3x_1+2x_2\le 2(x_1+x_2)+x_1\le 13.$

Given $A=(a_{ij})\in\mathbb{R}^{m\times n}$ $b\in\mathbb{R}^m$. assign y_i to each constraint

$$\max\ c_1x_1+c_2x_2+\cdots+c_nx_n.$$

$$\begin{array}{llr} \text{s.t.} & a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n\le b_1. & y_1 \\ & a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n\le b_2. & y_2 \\ & \vdots & \vdots \\ & a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n\le b_m & y_m. \end{array}$$

if $y_j\ge 0$. and $\sum y_i a_{ij}\ge c_j\ \ \forall j$. $\sum y_i b_i$ is an upper bound.

$\Rightarrow \max\ c_1x_1+c_2x_2+\cdots+c_nx_n\le \min\ y_1b_1+y_2b_2+\cdots+y_mb_m.$

Construct the LP: $\min\ y^Tb.$ s.t. $y^TA\ge c^T$ $y\ge 0.$

is the dual of primal LP $\max\ c^Tx$ s.t. $Ax\le b.$ $x\ge 0.$

Proposition. The dual of the dual is the primal.

Proof. Rewrite dual as $\max\ -b^Ty.$ s.t. $-A^Ty\le -c,\ y\ge 0.$

the dual of the dual is $\min\ -w^Tc.$ s.t. $-w^TA\ge -b^T.$ $w\ge 0.$ $\square$

Weak duality theorem primal feasible ←—— - - - - - - - - ——→ dual feasible.

If x is feasible for primal. y is feasible for dual. then $c^Tx \le y^Tb$

Proof. $c^Tx \le y^TAx \le y^Tb$. $(x \ge 0,\ y \ge 0)$. $\square$

Corollary. If primal has optimal z, dual has optimal w. $z \le w$.

(here for convention. let $\max \phi = -\infty$ and $\min \phi = \infty$).

Strong duality theorem. primal feasible ←——•——→ dual feasible.

If primal has finite optimal x^*. so is dual. and $c^Tx^* = y^{*T}b$.

primal \ dual	unbounded	infeasible	feasible
unbounded	✗	✓	✗
infeasible	✓	✓	✗
feasible.	✗	✗	✓

weak duality : unbounded $\Rightarrow$ infeasible. others are impossible.

strong duality : feasible $\Rightarrow$ feasible. but both infeasible ?

Primal: $\max\ 2x_1 - x_2$. s.t. $x_1 - x_2 \le 1$. $-x_1 + x_2 \le -2$. $x_1, x_2 \ge 0$.

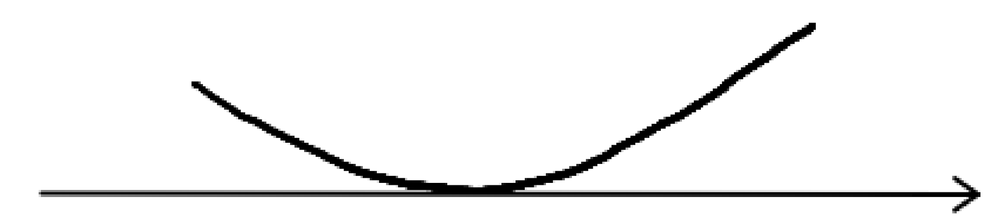
dual: $\min\ y_1 - 2y_2$ s.t. $y_1 - y_2 \ge 2$. $-y_1 + y_2 \ge -1$. $y_1, y_2 \ge 0$

Proof of strong duality : review of Farkas' lemma.

let $A \in \mathbb{R}^{m\times n}$. $b \in \mathbb{R}^m$. then exactly one of the followings is true.

① $\exists x \in \mathbb{R}^n$ s.t. $Ax = b$. $x \ge 0$. ② $\exists y \in \mathbb{R}^m$ s.t. $A^Ty \ge 0$. $b^Ty < 0$.

proof of Farkas' lemma: separating hyperplane (strict).

problem: closed set. conterexample: 

claim: conic combination of finite points is a closed set

① and ② cannot be both true: consider $y^T A x \ ? \ 0$.

if ① is not true. let $\alpha_1, \dots, \alpha_n$ be columns of A.

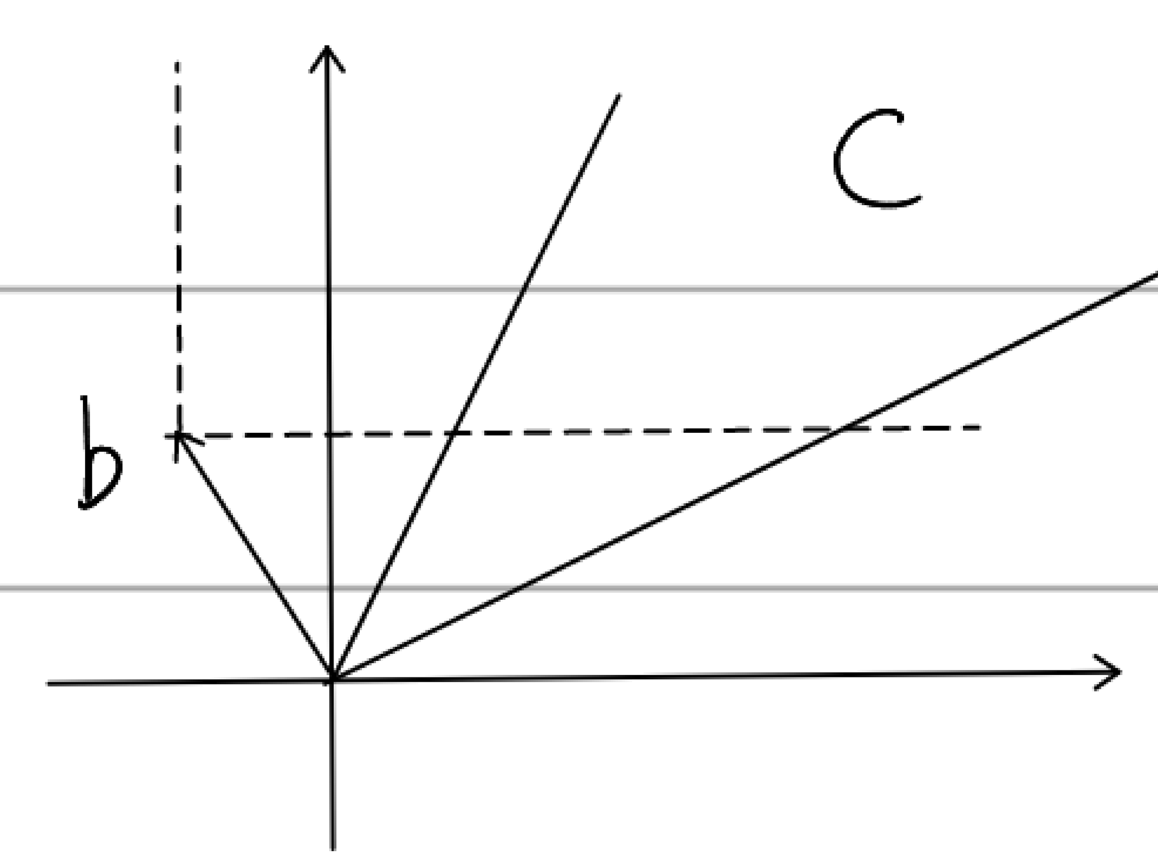
$C = \text{cone}(\alpha_1, \dots, \alpha_n)$. then $\exists\, w \neq 0, \beta$. s.t. $w^T b < \beta$. $w^T c > \beta$.

note that if $v \in C$, $\lambda v \in C \ \forall \lambda > 0 \Rightarrow \lambda w^T v > \beta$

$0 \in C \Rightarrow \beta < 0$. $w^T v > \beta / \lambda \ \forall \lambda > 0 \Rightarrow w^T v \geq 0$. $w^T b < \beta < 0$.

Corollary of Farkas' lemma: let $A \in \mathbb{R}^{m \times n}$. $b \in \mathbb{R}^m$

exactly one of the followings is true.



① $\exists\, x \in \mathbb{R}^n$. s.t. $Ax \geq b$, $x \geq 0$.

② $\exists\, y \in \mathbb{R}^m$. s.t. $A^T y \geq 0$. $b^T y < 0$. $y \leq 0$.

Proof: let $A' = (A, -I) \in \mathbb{R}^{m \times (m+n)}$. Apply Farkas' lemma on A', b.

① $\exists\, x' \in \mathbb{R}^{m+n}_{\geq 0}$ s.t. $A'x' = b$ iff $\exists\, x \in \mathbb{R}^n_{\geq 0}$ s.t. $Ax \geq b$. slack.

②. $\exists\, y \in \mathbb{R}^m$. s.t. $A'^T y \geq 0$ and $b^T y < 0 \Longleftrightarrow A^T y \geq 0$. $y \leq 0$. $b^T y < 0$

Now we are ready to prove strong duality theorem. (both feasible) let $\bar{x}$ and $\bar{y}$ be a feasible solution to primal and dual respectively.

weak duality $\Rightarrow c^T\bar{x} \le \bar{y}^T b$. dual is bounded. let γ be optimal.

suppose strong duality is not true. then $c^T\bar{x} < \gamma$, $\forall$ feasible $\bar{x}$.

$\Rightarrow \nexists x$, s.t. $Ax \le b$, $c^T x \ge \gamma \iff \begin{pmatrix} -A \\ c^T \end{pmatrix} x \ge \begin{pmatrix} -b \\ \gamma \end{pmatrix}$

Apply Farkas' lemma. ① does not hold $\Rightarrow$ ② $\exists y, w \le 0$ s.t.

$\begin{pmatrix} -A \\ c^T \end{pmatrix}^T \begin{pmatrix} y \\ w \end{pmatrix} \ge 0$ and $\begin{pmatrix} -b \\ \gamma \end{pmatrix}^T \begin{pmatrix} y \\ w \end{pmatrix} < 0$. Here are two cases. $w \overset{=}{<} 0$.

Case 1. $w = 0$. then $(-A)^T y \ge 0$, $(-b)^T y < 0$ and $y \le 0$.

Apply Farkas' lemma on $-A$ and $-b$. ② true $\Rightarrow$ ① false.

$\Rightarrow \nexists x$, s.t. $-Ax \ge -b$, $x \ge 0$. $\Rightarrow$ primal is infeasible.

Case 2. $w < 0$. dividing w on both sides, then we have.

$(-A^T, c) \begin{pmatrix} y/w \\ 1 \end{pmatrix} \le 0$. and $(-b^T, \gamma) \begin{pmatrix} y/w \\ 1 \end{pmatrix} > 0$.

$\Rightarrow A^T(y/w) \ge c$ and $b^T(y/w) < \gamma$. $\Big\}$ γ is optimal

$\Downarrow$ (y/w) is feasible for dual since $y \le 0$. $\Big\}$ contradiction

$\square$

Corollary (Complementary Slackness). x, y feasible for (P), (D).

x, y optimal iff $y^T(b - Ax) = 0$. and $x^T(A^T y - c)$.

Proof: $c^T x \le y^T A x \le y^T b$.

$\begin{cases} \text{either } y_i = 0, \text{ or } (Ax)_i = b_i \\ \text{either } x_j = 0 \text{ or } (A^T y)_j = c_j \end{cases}$