Lecture 3. Analysis on normed linear space (II)

saddle point. 鞍点.

interior point. 内点. boundary point 边界

$\exists \varepsilon > 0.\quad B(x, \varepsilon) \triangleq \{ y : \|x - y\| < \varepsilon \} \subseteq X.$

First-order necessary condition. 有偏导不够. $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} \\ 0. \end{cases}$

if $f(x^*)$ local minimum. f differentiable at x^*.

for any feasible direction v. $(\exists \varepsilon > 0 \mid \forall\, 0 < \delta < \varepsilon.\ x^* + \delta v \in X)$.

$v^T \nabla f(x^*) \geqslant 0.$

有方向导数不够

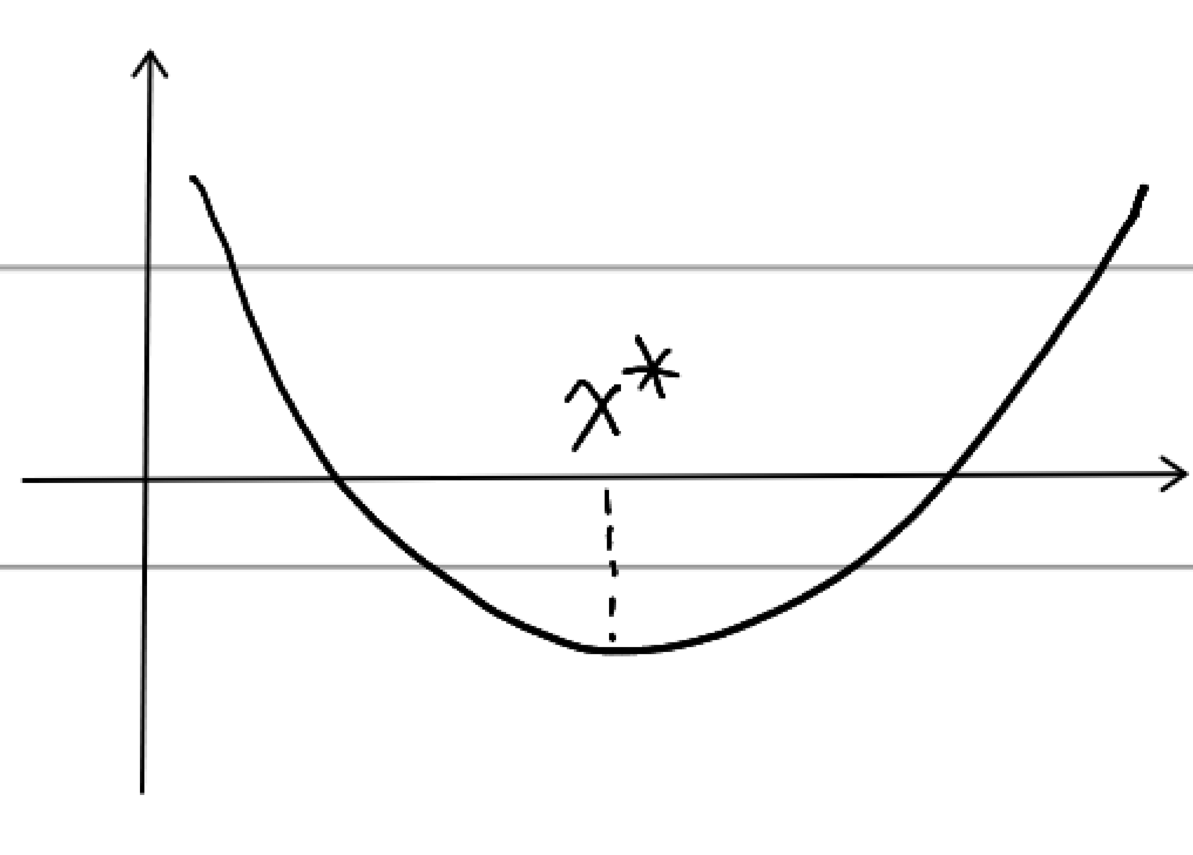
$f(x,y) = \begin{cases} \frac{y^2}{x} & x \neq 0 \\ 0 \end{cases}$

Cor. if x^* interior. $\nabla f(x^*) = 0$

偏导连续.

However. this condition does not suffice.

$f(x) = x^3$. $\quad f(x_1, x_2) = x_1^2 - x_2^2$. $\quad f(x_1, x_2) = x_1^3 - x_2^3$.



$x \uparrow x^*$: $f(x) \downarrow$ $\quad f'(x) < 0$.

$x \downarrow x^*$: $f(x) \downarrow$ $\quad f'(x) > 0$.

$f''(x) > 0$?

优化 $f: \mathbb{R}^n \to \mathbb{R}$.
that's why we
introduce Jacobian.

Second order partial derivative. $\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)$

Hessian matrix. denoted by $\nabla^2 f(x_0)$.

Hessian is given by $[\nabla^2 f(x_0)]_{ij} = \frac{\partial^2 f(x_0)}{\partial x_i \partial x_j}$.

if $\frac{\partial f}{\partial x_i \partial x_j}$ and $\frac{\partial f}{\partial x_j \partial x_i}$ exists in $B(x, \varepsilon)$.

and continuous at x_0. then $\frac{\partial f}{\partial x_i \partial x_j} = \frac{\partial f}{\partial x_j \partial x_i}$.

Cor. Hessian is a symmetric matrix.

Taylor expansion (second order).

$$f(x_0+\delta) = f(x_0) + \delta f'(x_0) + \frac{\delta^2}{2} f''(x_0) + o(\delta^2).$$

$$f(\vec{x}_0+\vec{\delta}) = f(\vec{x}_0) + \nabla f(\vec{x}_0)^T \vec{\delta} + o(\|\delta\|).$$

$$f(\vec{x}_0+\vec{\delta}) = f(\vec{x}_0) + \nabla f(\vec{x}_0)^T \vec{\delta} + \frac{1}{2} \vec{\delta}^T \nabla^2 f(\vec{x}_0) \vec{\delta} + o(\|\delta\|^2)$$

or expansion.

$$f(\vec{x}_0+\vec{\delta}) = f(\vec{x}_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x}_0)\, \delta_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x}_0)\, \delta_i \delta_j + o(\ ^2)$$

Example. $f(x) = w^T x + b$. $\nabla f(x) = w$ $\nabla^2 f(x) = 0$.

$f(x) = x^T A x$ $\nabla f(x) = (A+A^T)x$ $\nabla^2 f(x) = A+A^T$

in particular. if A symmetric. $\nabla f(x) = 2Ax$. $\nabla^2 f(x) = 2A$.

verify it: $f(x_0+\delta) = (x_0+\delta)^T A (x_0+\delta) = x_0^T A x_0 + \delta^T A x_0 + x_0^T A \delta + \delta^T A \delta$

$= f(x_0) + (x_0^T A + x_0^T A^T)\delta + \delta^T A \delta$

Taylor: $f(x_0+\delta) = f(x_0) + (Ax_0 + A^T x_0)^T \delta + \frac{1}{2}\delta^T (A+A^T)\delta + o(\|\delta\|^2)$.

Chain rule for Hessian.

$h(x) = g(f(x))$ $\quad Dh(x) = Dg(f(x)) \cdot Df(x)$

if $h: \mathbb{R}^n \to \mathbb{R}^m$. $\quad Dh: \mathbb{R}^n \to \mathbb{R}^{m\times n}$ what is $D^2 h$?

$h(x): \mathbb{R}^n \to \mathbb{R}$. $\quad g(x): \mathbb{R}^n \to \mathbb{R}$. $\quad f(x) = Ax + b$

$\nabla h(x) = \nabla g(f(x)) \cdot A$ $\quad = 0$.

$\nabla^2 h(x) = \nabla g(f(x)) \cdot DA + A^T D(\nabla g(f(x)))$

$= A^T \nabla^2 g(f(x)) A$.

In particular. $g(t) = f(x_0 + tv)$ $\quad \nabla^2 g(t) = v^T \nabla^2 f(x_0 + tv) v^T$.

if $f(x_0)$ is a local minimum. we know $\nabla f(x_0) = 0$

$f(x_0 + \delta) \geq f(x_0)$ $\quad \Longrightarrow \delta^T \nabla^2 f(x_0 + \delta) \delta \geq 0$.

忽略小o项

Definite matrix. Second-order necessary condition

Semidefinite. positive semidefinite if

$A^T = A$ (symmetric) $\quad \forall x.$ $\quad x^T A x \geq 0$

positive definite if symmetric and $\forall x \neq 0$ $\quad x^T A x > 0$

negative if ≤ 0 (semidefinite) $\quad < 0$ (definite)

indefinite if $x_1^T A x_1 < 0 < x_2^T A x_2$

Remark. x^TAx. quadratic form 二次型

$$x^TAx = x^TA^Tx = x^T\left(\tfrac{1}{2}(A+A^T)\right)x$$

Properties of definiteness.

- $A \geq 0$ (semidefinite) iff all eigenvalues $\lambda \geq 0$.
- $A > 0$ (definite) iff all eigenvalues $\lambda > 0$

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad x = (a, b)^T. \qquad \qquad a^2+b^2+(a-b)^2$$

$$x^TAx = (2a-b, -a+2b)\begin{pmatrix} a \\ b \end{pmatrix} = 2a^2 - ab - ab + 2b^2 \geq 0.$$

$$\det(\lambda I - A) = (\lambda-2)^2 - 1 = 0 \Rightarrow \lambda = 1, 3.$$

why? $\quad A = U\Lambda U^T = \sum_{i=1}^n \lambda_i v_i v_i^T$

$$U = (v_1, v_2, \cdots v_n) \qquad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots \lambda_n).$$

$$x^TAx = (U^Tx)^T\Lambda(U^Tx) = \sum_{i=1}^n \lambda_i \|v_i x\|^2.$$

$$x = \sum_{i=1}^n y_i v_i \qquad x^TAx = \sum_{i=1}^n \lambda_i \|y_i v_i\|^2.$$

Proof of $\delta^T \nabla^2 f(x_0)\delta \geq 0$:

otherwise $\exists \lambda < 0$. let v be the eigenvector with respect to λ.

$$f(x_0+tv) = f(x_0) + \nabla f(x_0)^T(tv) + \frac{1}{2}(tv)^T\nabla^2 f(x_0)(tv) + o(\|tv\|^2)$$

$$= f(x_0) + \frac{\lambda}{2}t^2\|v\|^2 + o(t^2\|v\|^2).$$

$\exists \varepsilon > 0$. if $|t| < \varepsilon$. $o(t^2\|v\|^2) < \frac{\lambda}{4} t^2 \|v\|^2$.

Thus. $f(x_0 + tv) < f(x_0) + \frac{\lambda}{4} t^2 \|v\|^2 < f(x_0)$.

sufficient ? $f(x) = x^3$. $f''(0) = 0$. $\neq 0$.

positive definite suffices. $\delta^T \nabla^2 f(x_0) \delta > 0 \quad \forall \delta \in \mathbb{R}^n$.

Proof: given $u \neq 0$. $f(x_0 + tu) = f(x_0) + \frac{t^2}{2} u^T \nabla^2 f(x_0) u + o(t^2\|u\|^2)$.

$u = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$. $\lambda_{min} = \min\{\lambda_1, \ldots \lambda_n\}$.

$u^T \nabla^2 f(x_0) u = \sum_{i=1}^{n} a_i^2 \|v_i\|^2 . \lambda_i \geq \lambda_{min} \|u\|^2$.

$\exists \varepsilon > 0$. if $|t| < \varepsilon$. $o(t^2\|u\|^2) < \frac{\lambda_{min}}{4} t^2 \|u\|^2$ $\square$

More about definiteness.

given matrix $A \in \mathbb{R}^{n \times n}$. $= \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

a $k \times k$ principal submatrix. is a submatrix of A

consisting of k rows and k columns. with same indices

$I = \{i_1, i_2, \ldots i_k\}$. $A_I = \begin{pmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \cdots & a_{i_1 i_n} \\ \vdots & & & \\ a_{i_n i_1} & a_{i_n i_2} & - & a_{i_n i_n} \end{pmatrix}$

principal minor 主子式 $|A_I|$. $\det A_I$.

leading principal minor 顺序主子式 $I = [k]$.

Sylvester's criterion.

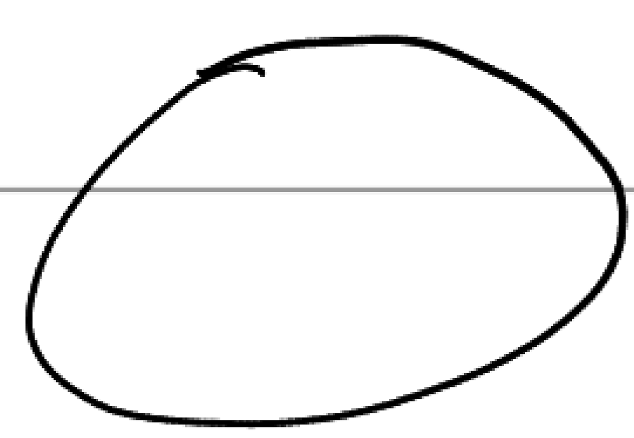
- $A \succ 0$ iff $D_k(A) \triangleq \det A_{[k]} > 0$ for $k = 1, 2, \cdots n$.
- $A \succeq 0$ iff $\det A_I \geq 0$ for all $I \subseteq \{1, 2, \cdots, n\}$.
- $A \succeq 0$ if $D_k(A) > 0$ for all $k = 1, 2, \cdots n-1$. $\det A \geq 0$

$A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$. counterexample if only $D_k(A) \geq 0$.

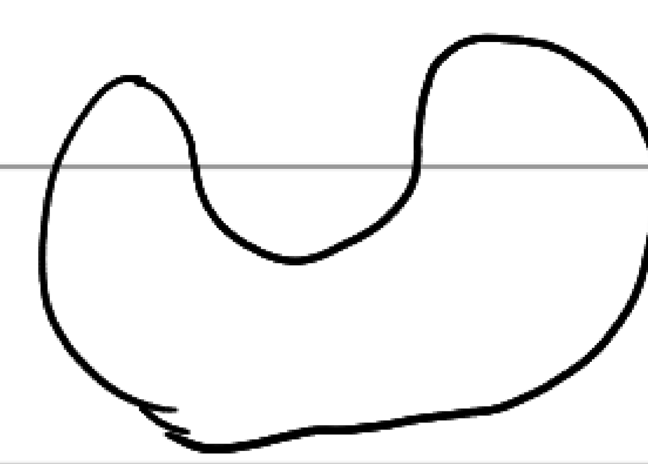
Line. $y = kx + b$.

given x, y. $z = y + \theta(x-y) = \theta x + (1-\theta) y$. $\theta \in \mathbb{R}$.

Convex set. S convex if $\forall x, y \in S$. $\underset{\in [0,1]}{\theta} x + (1-\theta) y \in S$.



convex



non convex.

凹 is not convex.

Intersection of convex sets.

$\{C_i : i \in I\}$ is a family of convex sets.

$\bigcap_{i \in I} C_i$ is also convex. proof is trivial.