Lecture 4. Geometry: affine and convex sets.

Heine-Borel Theorem: $\mathbb{R}^n$. bounded closed $\Leftrightarrow$ compact.

line: $z = x + \theta(y-x) = \theta y + (1-\theta)x$. $\theta \in \mathbb{R}$.

affine set: $\forall x, y \in S$. $\forall \theta$. $z \in S$.

affine combination
given $x_1 \cdots x_n \in S$.
$\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n$.
s.t. $\theta_1 + \theta_2 + \cdots + \theta_n = 1$.

Example: solution set of linear equations

$\{x : Ax = b\}$.

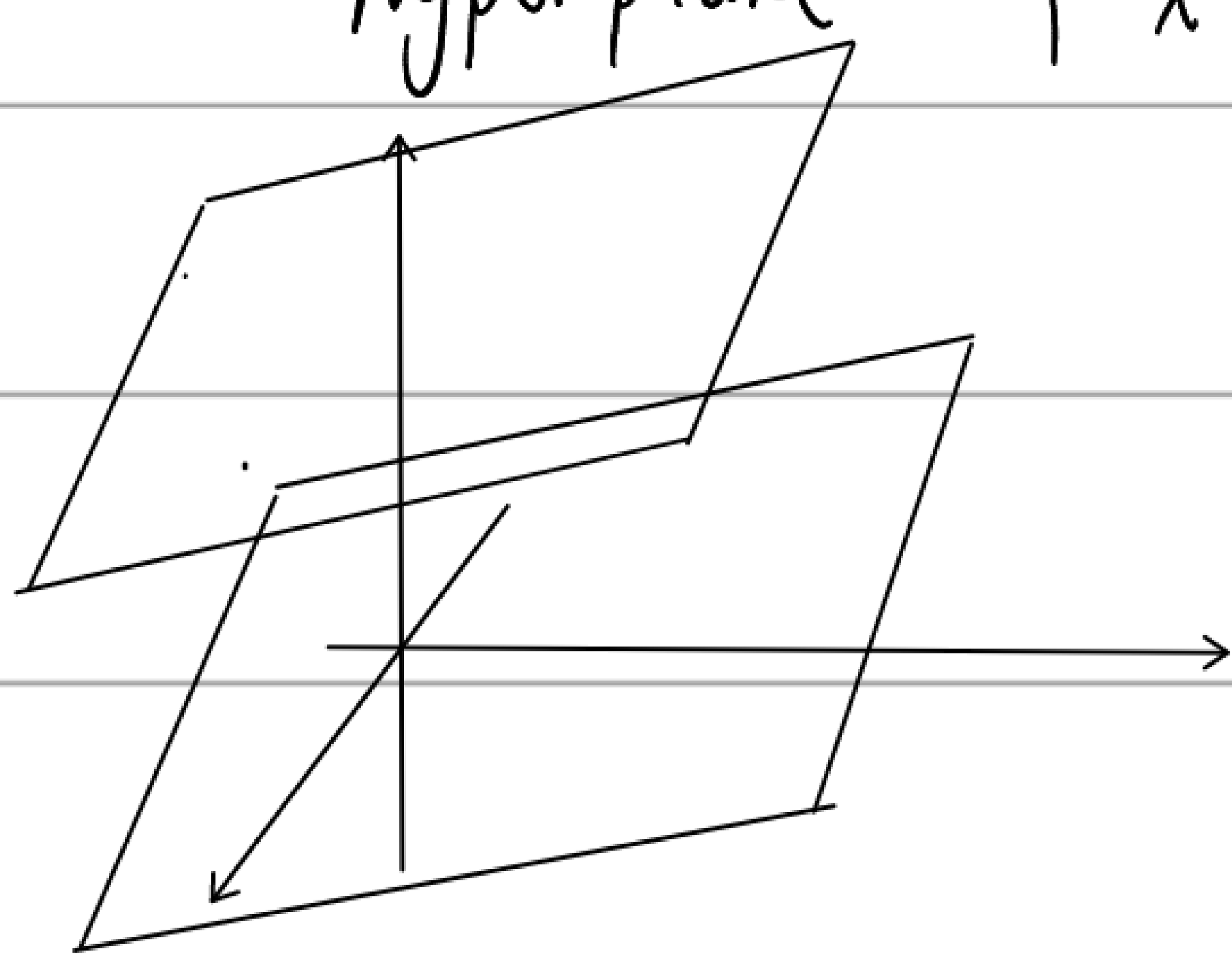
why affine?
$S - x_0$ linear space.
so linear + offset.

conversely. every affine set can be expressed as solution set of linear equations.

in particular. If $A \in \mathbb{R}^{1\times n}$

$x+b$. $y+b \in S$.
$\alpha x + \beta y + b$
$= \alpha(x+b) + \beta(y+b) + (1-\alpha-\beta)b$
$\in S$.

hyperplane: $\{x : w^T x = b\}$. $w \neq 0$.



affine set is an intersection of finite hyperplane

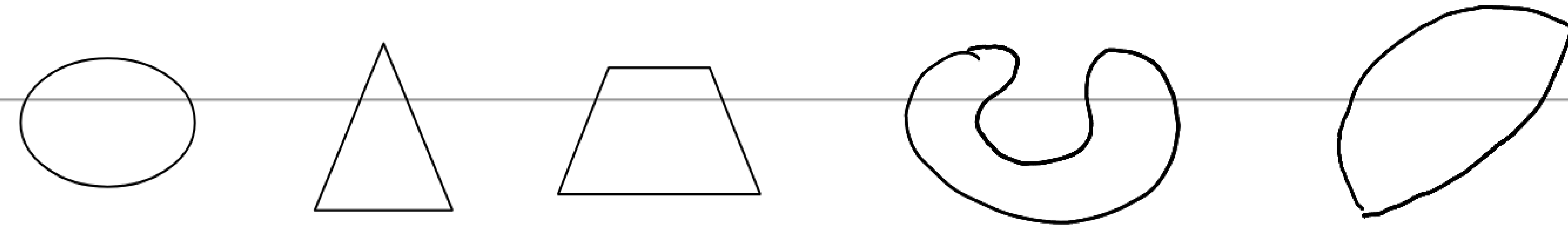
if $0 \in$ hyperplane. a $(n-1)$-dim subspace.

If $A \neq 0$ in affine set. $\Rightarrow$ intersection of finite hyperplane.

segment: $z = x + \theta(y-x) = \theta y + (1-\theta)x$. $\theta \in [0,1]$

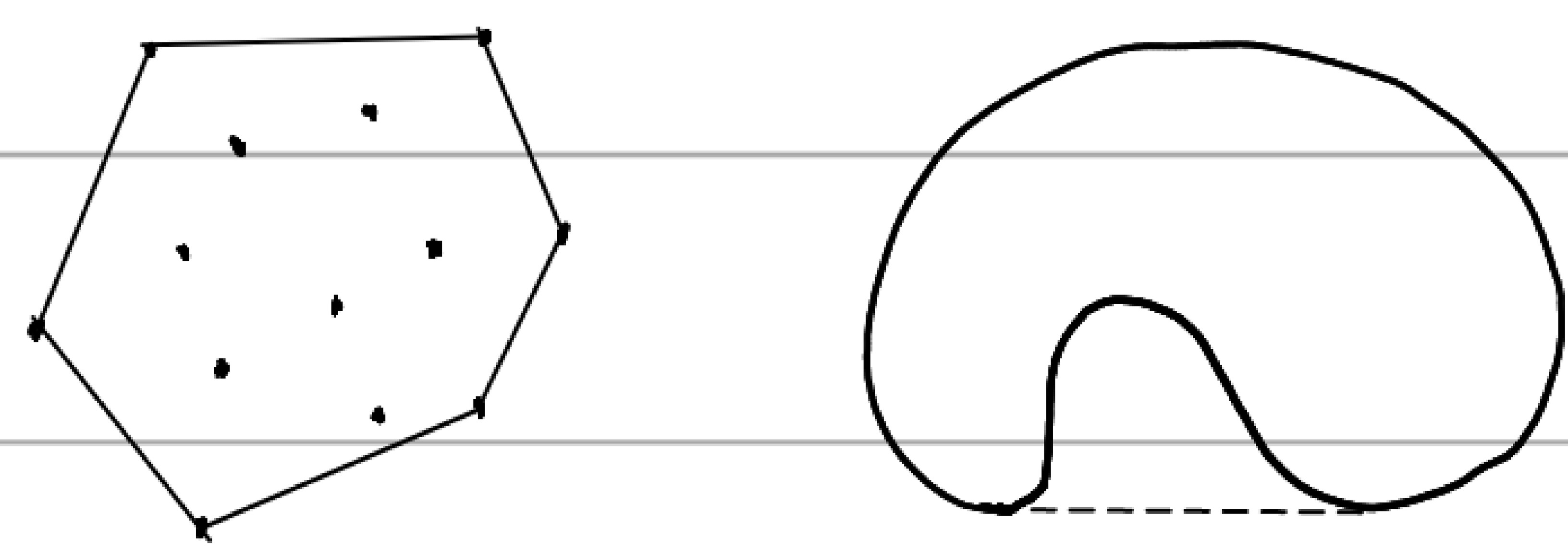
convex set: $\forall x, y \in S$. $\forall \theta \in [0,1]$. $z \in S$.

 is not convex.



convex combination: $\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n$. $\theta_i \geq 0$. $\Sigma \theta_i = 1$.

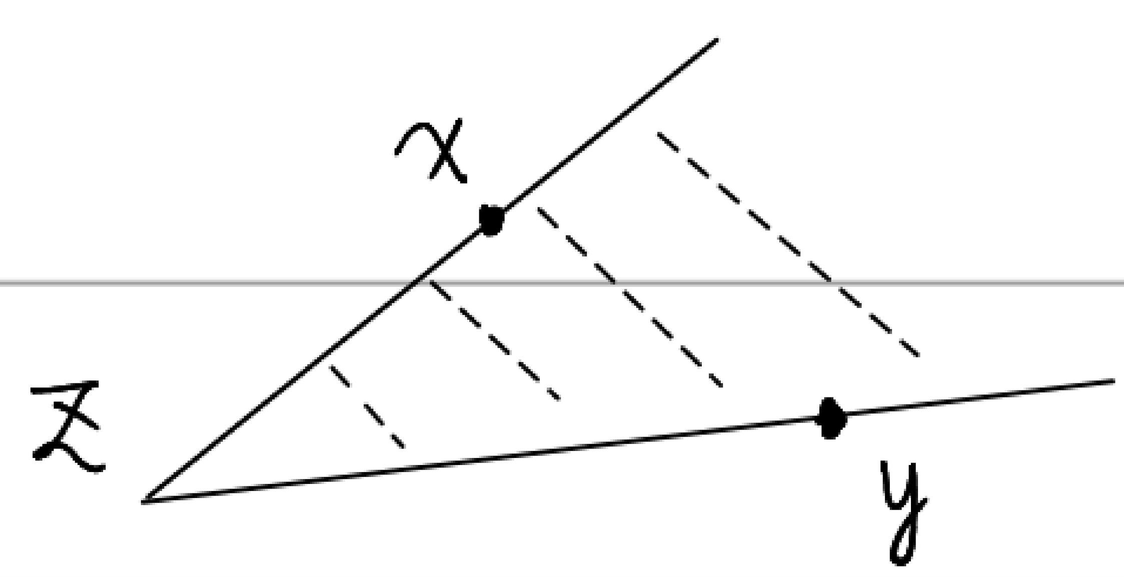
S is convex iff S contains every convex combination of its points.



convex hull: set of all convex combination of points in S.

$\{ \Sigma \theta_i x_i : \theta_i \geq 0.\ \Sigma \theta_i = 1.\ x_i \in S \}$.

conic combination. $z = \theta_1 x + \theta_2 y$. $\theta_1 . \theta_2 \geq 0$.



convex cone: set that contains all conic combination of points in S.

Some examples of convex sets:

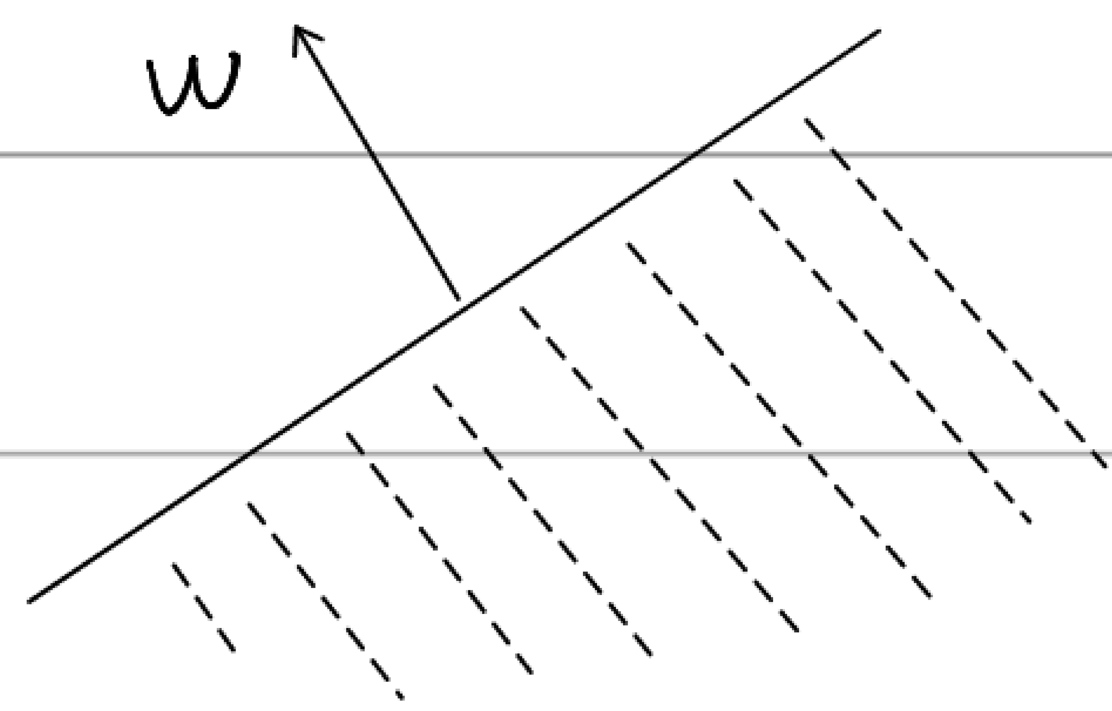
$\mathbb{R}^n$. affine sets. hyperplanes.

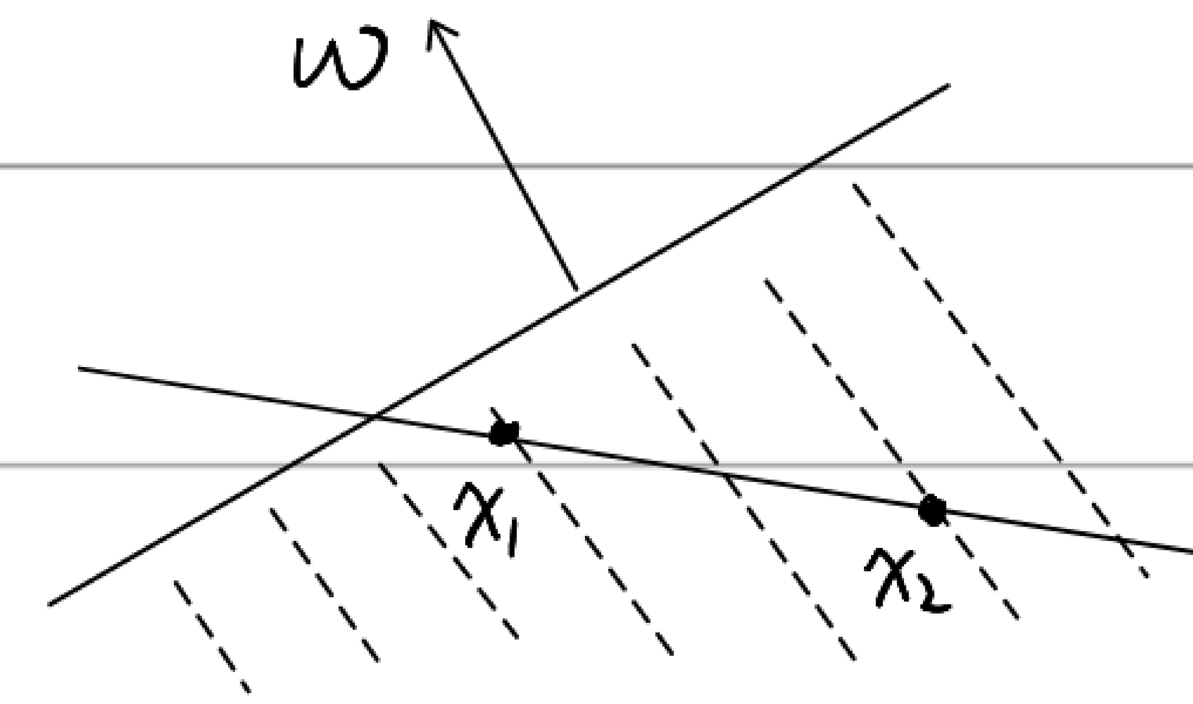
half spaces: a hyperplane divide $\mathbb{R}^n$ into 2 halfspaces.

$\{x : w^T x = b\}$. $\{x : w^T x < b\}$. $\{x : w^T x > b\}$

closed halfspace: $\{x : w^T x \leq b\}$ $w \neq 0$.

halfspaces are convex, but not affine.





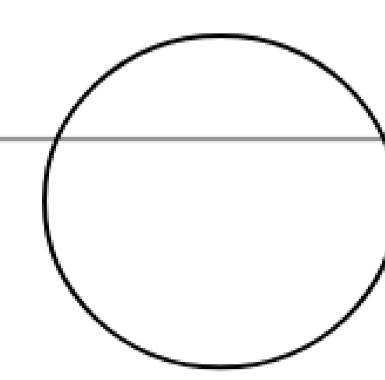
open halfspace. $\{x: w^T x < b\}$ interior points of closed.

$\bar{\theta} \triangleq 1-\theta$. $\forall x, y \in S = \{x: w^T x \leq b\}$.

$$w^T(\theta x + \bar{\theta} y) = \theta w^T x + \bar{\theta} w^T y \leq \theta \cdot b + \bar{\theta} \cdot b = b.$$

Euclidean balls and ellipsoids.

Euclidean ball:



$\{x: \|x - x_0\|_2 \leq r\}$.

$\iff \{x: x = x_0 + r d.\ \|d\|_2 \leq 1\}$.

triangle inequality.

$$\|\theta x + \bar{\theta} y - x_0\|_2 = \|\theta(x - x_0) + \bar{\theta}(y - x_0)\|_2 \leq \cdots$$

norm ball: $\{x: \|x - x_0\| \leq r\}$.

Ellipsoid: $\left\{x: \frac{x_1^2}{\lambda_1^2} + \frac{x_2^2}{\lambda_2^2} \leq 1\right\}$. convex.

Proof: $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2\}$. $E = \{\Lambda u: \|u\|_2 \leq 1\}$

suppose $x_i = \Lambda u_i$. $\theta x_1 + \bar{\theta} x_2 = \Lambda(\theta u_1 + \bar{\theta} u_2) = \Lambda u$

for some $\|u\|_2 \leq 1$.

$E = \{x: \|\Lambda^{-1} x\|_2 \leq 1\} = \{x: x^T \Lambda^{-2} x \leq 1\}$

in general. $E = \{ x_0 + \Lambda u : \|u\|_2 \le 1 \} = \{ x : (x-x_0)^T \Lambda^{-2} (x-x_0) \le 1 \}$.

shift

with rotation. $A = Q \Lambda Q^T$. Q is orthogonal. $A \succ 0$

positive definite.

$$E = \{ x_0 + Au : \|u\|_2 \le 1 \} = \{ x : (x-x_0)^T A^{-2} (x-x_0) \le 1 \}.$$

Proposition: the image of a convex set under an affine function is also convex.

proof: $f(x) = Ax + b$ is affine.

$C \subseteq \mathbb{R}^n$ is convex. give x_1, x_2. and $y_i = f(x_i)$

goal: $\forall \theta \in [0,1]$. $\theta y_1 + \bar\theta y_2 \in f(C)$.

$$\theta y_1 + \bar\theta y_2 = (\theta A x_1 + \theta b) + (\bar\theta A x_2 + \bar\theta b) = A(\underbrace{\theta x_1 + \bar\theta x_2}_{\in C}) + b.$$

The inverse image of a convex set is also convex.

A nongeometric example: positive semidefinite matrices.

The set of positive semidefinite matrices is convex.

$S_+^n \triangleq \{ A \in \mathbb{R}^{n \times n} : A \succeq 0 \}$. S_{++}^n positive definite.

proof: 1. symmetric. $A. B.$ symmetric $\Rightarrow \theta A. \bar\theta B.$ symmetric.

2. $x^T(\theta A + \bar\theta B) x \geqslant 0$

$\quad\quad\quad \|$

$\theta x^T A x + \bar\theta x^T B x \geqslant 0$

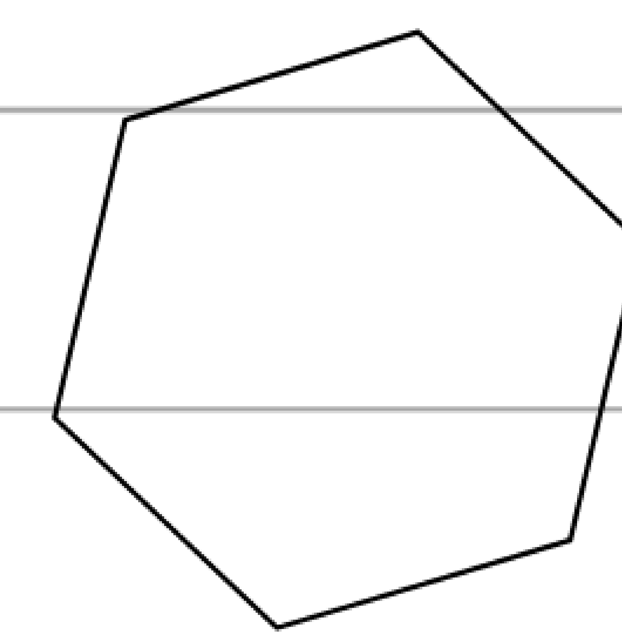
Intersection: $\{C_i : i \in I\}$ a family of convex sets. $\bigcap_i C_i$ convex.

Polyhedron / Polyhedra 多面体.

a polyhedron is defined as the solution space of a finite number of linear inequalities. or. intersection of halfspaces

$P = \{x : w_i^T x \le b_i,\ i = 1, 2, \dots, m\}$.

affine sets, halfspaces are all polyhedron.



Polyhedra are all convex.

Polytope 多胞体. bounded polytope.

Simplex / simplices or simplexes. 单纯形.

so-named because it represents the simplest polytope.

0 - simplex : point   1- simplex : line segment.

2 - simplex : triangle   3- simplex : tetrahedron.

k-simplex is the convex hull of k+1 affinely independent points.

$S = \{\theta_0 u_0 + \dots + \theta_k u_k : \sum \theta_i = 1,\ \theta_i \ge 0\}$

$w_0 x_0 + w_1 x_1 + \dots + w_k x_k = b$.

$x_1 - x_0,\ x_2 - x_0,\ \dots,\ x_k - x_0$ linearly independent

standard simplex: $u_0 \dots u_k$ unit vector $S = \{x : x_0 + \dots + x_k = 1\}$

let $y \triangleq (\theta_1, \dots \theta_k) \in \mathbb{R}^k$ $B = (u_1 - u_0, \dots u_k - u_0) \in \mathbb{R}^{n \times k}$ has rank k.

$S = \{u_0 + By : \sum y_i \le 1,\ y_i \ge 0\}$

B has rank k $\Rightarrow$ $\exists$ nonsingular $A = (A_1, A_2) \in \mathbb{R}^{n\times n}$ s.t.

$$A^T B = \begin{pmatrix} A_1^T \\ A_2^T \end{pmatrix} B = \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

$$\underset{x:\, x=}{S = \{ u_0 + By \}} \qquad A^T x = A^T u_0 + \begin{pmatrix} y \\ 0 \end{pmatrix} \qquad \begin{aligned} A_1 x &= A_1 u_0 + y \\ A_2 x &= A_2 u_0. \end{aligned}$$

so $x \in S$ iff $A_2 x = A_2 u_0$ and $y = A_1 u_0 - A_1 x$ satisfy $\begin{cases} y_i \geq 0 \\ \sum y_i = 1 \end{cases}$

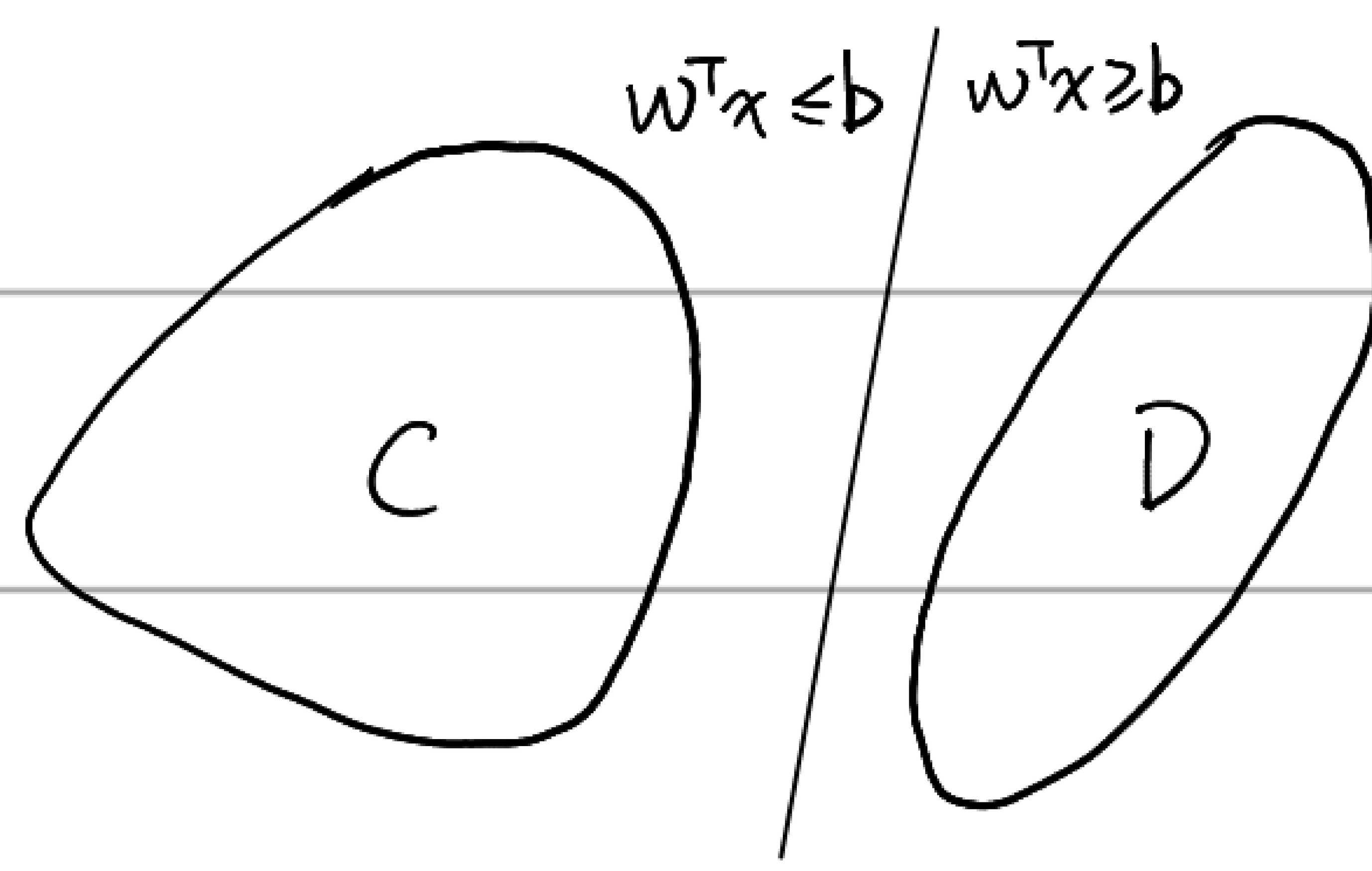
$\sum y_i = \mathbf{1}^T y$. so. $A_1 x \geq A_1 u_0$ and. $1^T A_1 x \leq 1^T A_1 u_0 + 1$.

Separating hyperplane. : separate convex sets that do not intersect.

separating hyperplane theorem:

suppose C. D are two convex sets that do not intersect.

Then. $\exists\, w \neq 0$. and b. s.t. $\begin{matrix} w^T x \leq b \\ w^T x \geq b \end{matrix}$ for $\begin{matrix} x \in C \\ x \in D. \end{matrix}$



strict separation.

if $\begin{matrix} w^T x < b \text{ for } x \in C \\ w^T x > b \text{ for } x \in D. \end{matrix}$

supporting hyperplane. $\partial S \triangleq$ boundary of S.

If $w \neq 0$. satisfies $w^T x \leq w^T x_0$ for all $x \in S$.

$\{x : w^T x = w^T x_0\}$ is called a supporting hyperplane to S at x_0.

separating S and $\{x : w^T x > w^T x_0\}$