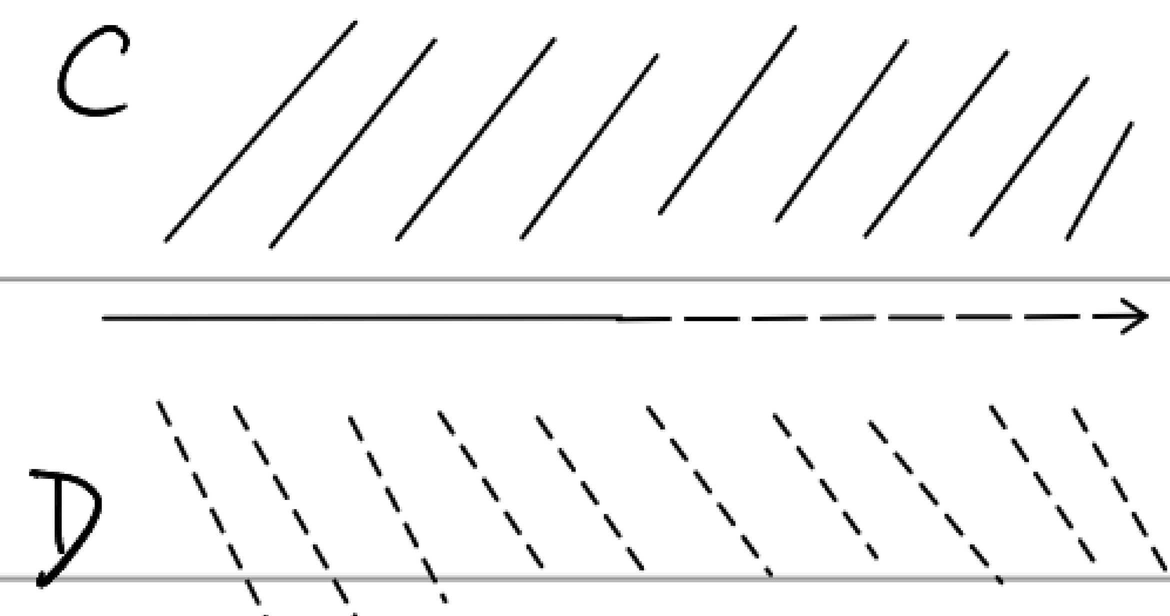
Lecture 5. Geomerty: separating and supporting hyperplane.

Proof of separating hyperplane theorem.

strict separating theorem.



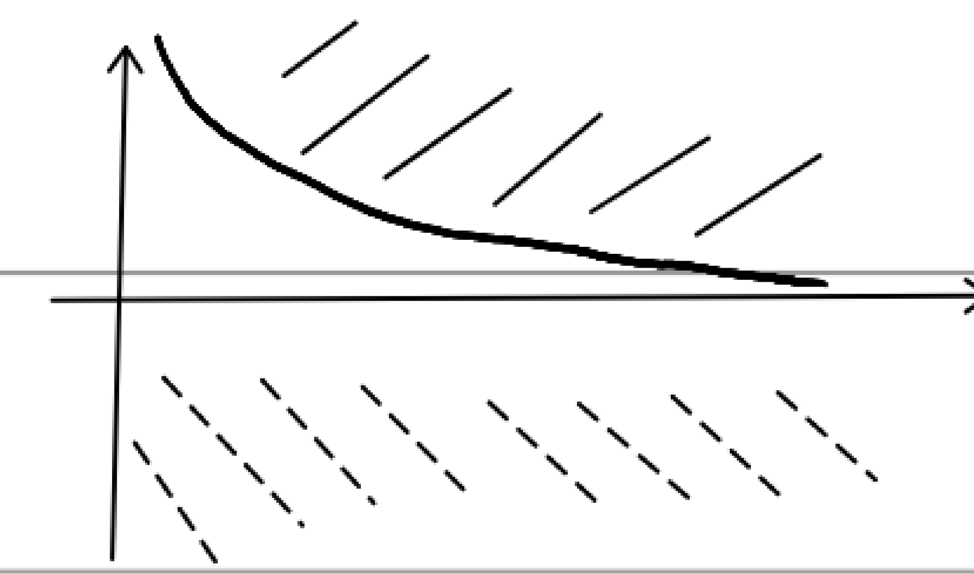
C. D are two closed convex sets and at least one of them is bounded.

then $\exists$ strict separating hyperplane. i.e.

$\exists$ w. b. s.t. $w^T x < b < w^T y \quad \forall x \in C. y \in D.$

counterexample.

two closed but unbounded set?



Proof of strict separating hyperplane theorem.

First. find $u \in C$, $v \in D$. s.t. dist (u,v) = dist (C,D).

where dist $(C,D) = \inf_{x \in C, y \in D} \|x-y\|$ why exists?

w.l.o.g. C bounded. given $u \in C$. $v \in D$.

$D \cap \{x : \|u-x\| \le \|u-v\|\}$ convex and compact.

$\Rightarrow \exists \bar{v} \in D$ $\underset{\to 0}{\text{dist}(u,\bar{v}) =}$ dist (u, D) — continuous function

C compact $\Rightarrow \exists \underset{u}{\arg\min}$ dist $(u, D) \triangleq \bar{u}$.

$\bar{u} \in C$, $\bar{v} \in D$ s.t. $\|\bar{u}-\bar{v}\| = \text{dist}(C, D)$.

let $w = \bar{v} - \bar{u}$, $b = \dfrac{\|\bar{u}\|^2 - \|\bar{v}\|^2}{2}$.

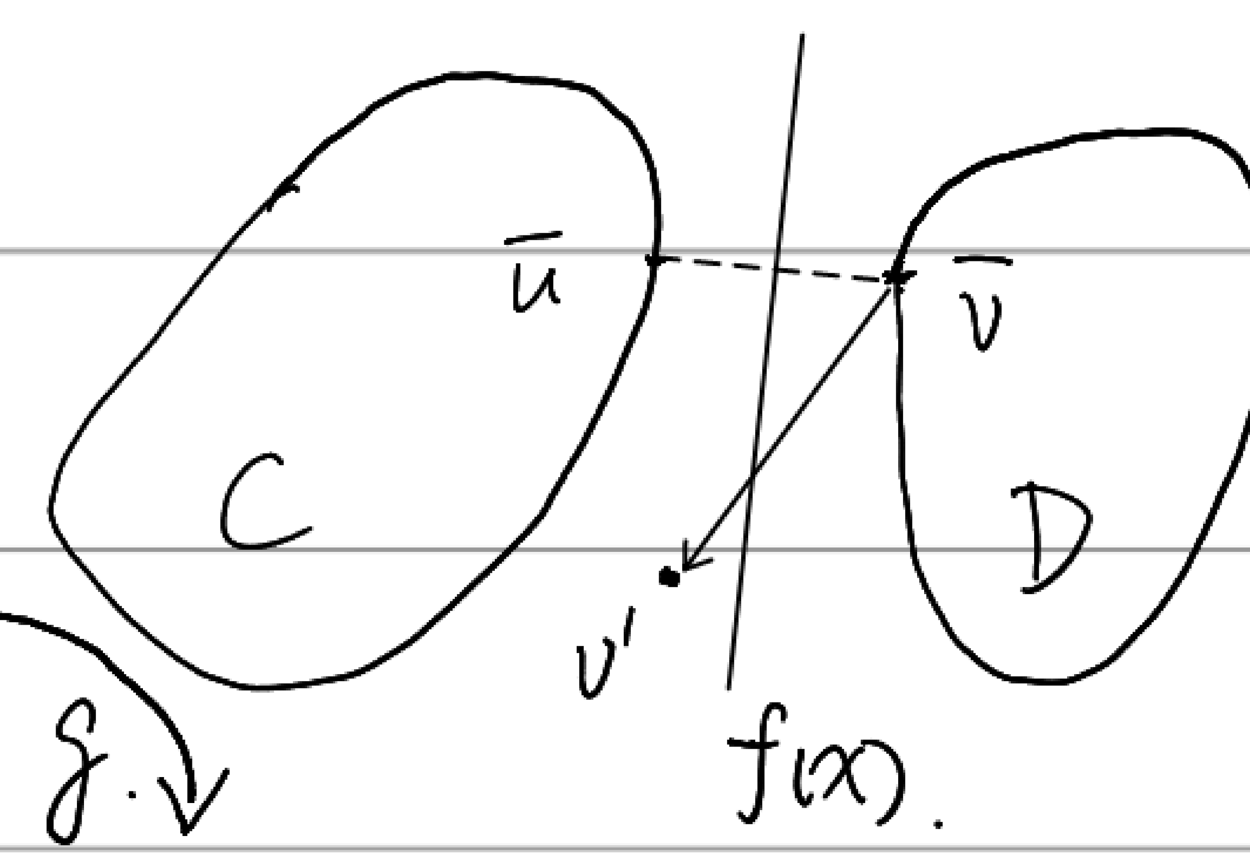
our separating hyperplane is $f(x) = w^T x + b$.

we claim that $f(x) > 0$, $\forall x \in D$ and $f(x) < 0$, $\forall x \in C$.

remark: why b is chosen? $f\left(\dfrac{\bar{u}+\bar{v}}{2}\right) = (\bar{v}-\bar{u})^T \left(\dfrac{\bar{u}+\bar{v}}{2}\right) + \dfrac{\|\bar{u}\|^2 - \|\bar{v}\|^2}{2} = 0$.

suppose for the sake of contradiction that $\exists v' \in D$ s.t. $f(v') \leq 0$.

$$\Rightarrow (\bar{v}-\bar{u})v' - \frac{\|\bar{v}\|^2 - \|\bar{u}\|^2}{2} \leq 0.$$



Define $g(x) = \|x - \bar{u}\|^2$.

claim: $v' - \bar{v}$ is a descent direction at $\bar{v}$ of g.

proof: $\nabla g(\bar{v})^T (v' - \bar{v}) = 2(\bar{v}-\bar{u})^T (v' - \bar{v})$

$= 2(-\|\bar{v}\|^2 + \bar{v}^T v' + \bar{u}^T \bar{v} - \bar{u}^T v')$.

$\nabla g(x) = \nabla (x-c)^T (x-c)$

$= \nabla (x^T x - 2c^T x + c^T c)$

$= 2(x-c)$

$= 2(-\|\bar{v}\|^2 + w^T v' + \bar{u}^T \bar{v})$

$\leq 2\left(-\|\bar{v}\|^2 - \dfrac{\|\bar{u}\|^2 - \|\bar{v}\|^2}{2} + \bar{u}^T \bar{v}\right)$ < 0

$= -(\|\bar{u}\|^2 + \|\bar{v}\|^2 - 2\bar{u}^T \bar{v}) = -\|\bar{u} - \bar{v}\|^2$

Thus $\exists \theta > 0$ s.t. $g(\bar{v} + \theta'(v' - \bar{v})) < g(\bar{v})$ for $\forall \theta' < \theta$.

by convexity. if $\theta < 1$ $\bar{v} + \theta(v' - \bar{v}) \in D \Rightarrow$ contradiction. $\square$

corollary 1. If $C \subseteq \mathbb{R}^n$ closed and convex. $d \in \mathbb{R}^n \setminus C$.

Then C and d can be strictly separated by a hyperplane.

corollary 2. Supporting hyperplane theorem.

suppose $C \in \mathbb{R}^n$. ball $B(x, \varepsilon) \triangleq \{ y \overset{\in \mathbb{R}^n}{:} \|y - x\| \leq \varepsilon \}$.

interior : $\text{int } C \triangleq \{ x \overset{\in C}{:} \exists \varepsilon > 0. \text{ s.t. } B(x, \varepsilon) \subseteq C \}$.

closure : $\text{cl } C \triangleq \{ x \overset{\in \mathbb{R}^n}{:} \exists \underbrace{x_0 \dots x_n, \dots}_{\in C} \text{ s.t. } \lim x_n \to x \}$.

(by definition) if C open $\text{int } C = C$. closed. $\text{cl } C = C$.

boundary : $\text{bd } C$. or $\partial C \triangleq \text{cl } C \setminus \text{int } C$.

SHT : $\forall C \neq \phi$ is convex. $x_0 \in \partial C$ $\exists w^T \neq 0$ s.t.

$\forall x \in C. \quad w^T x \leq w^T x_0$.

where $\{ x : w^T x = w^T x_0 \}$ is called a supporting hyperplane of C at x_0

Proof: If $\text{int } C = \phi$. C lies in an affine set of $\dim < n$.

otherwise $\exists$ $n+1$ affinely independents points in C

C contains an n-simplex and thus interior points.

if $\text{int } C \neq \phi$. let $C_\varepsilon \triangleq \{ x : B(x, \varepsilon) \subseteq \text{cl } C \}$.

$\forall \varepsilon > 0$. $\exists w_\varepsilon \neq 0$. s.t. $w_\varepsilon^T x < w_\varepsilon^T x_0$ for $\forall x \in C_{-\varepsilon}$.

w.l.o.g let $\|w_\varepsilon\| = 1$. let $\varepsilon_n = \frac{1}{n} \to 0$.

$\exists$ a subsequence of $\{w_{\varepsilon_n}\}$ converge to w.

$\forall x \in \text{int } C$. $\exists N > 0$. s.t $\forall n > N$. $w_{\varepsilon_n}^T x < w_{\varepsilon_n}^T x_0$

$\Rightarrow w^T x \leq w^T x_0$ $\quad \forall y \in \partial C$. $\exists \{x_n \in C\} \to y$.

$\Rightarrow w^T y \leq w^T x_0 \quad \Rightarrow \forall v \in C \quad w^T v \leq w^T x_0$. $\square$

Finally. prove separating hyperplane theorem.

$\forall C. D. \subseteq \mathbb{R}^n$. are two convex sets. $\exists w^T \neq 0$ and b. s.t.

$\forall u \in C. \quad \forall v \in D. \quad w^T u \leq b \leq w^T v$.

Proof: consider $C-D \triangleq \{u-v : u \in C, v \in D\}$.

$C-D$ is also convex. separating $C-D$ and $\{0\}$

Goal: find $w \neq 0$. s.t. $w^T x \leq 0$ for $\forall x \in C-D$.

Case 1: $0 \notin \partial(C-D)$ strictly separating hyperplane theorem.

$\exists w. \neq 0$ separating (strictly) $cl(C-D)$ and 0

$\Rightarrow \forall u \in C. \forall v \in D \quad w^T(u-v) \leq 0$

Case 2: $0 \in \partial(C-D)$. supporting hyperplane theorem.

$\exists w \neq 0 \quad \forall u \in C. \forall v \in D. \quad w^T(u-v) \leq 0$. $\square$

Application : Farkas lemma.

Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. then exactly one of the following sets must be empty :

1. $\{x : Ax = b,\ x \geq 0\}$.
2. $\{y : A^T y \leq 0.\ b^T y > 0\}$.

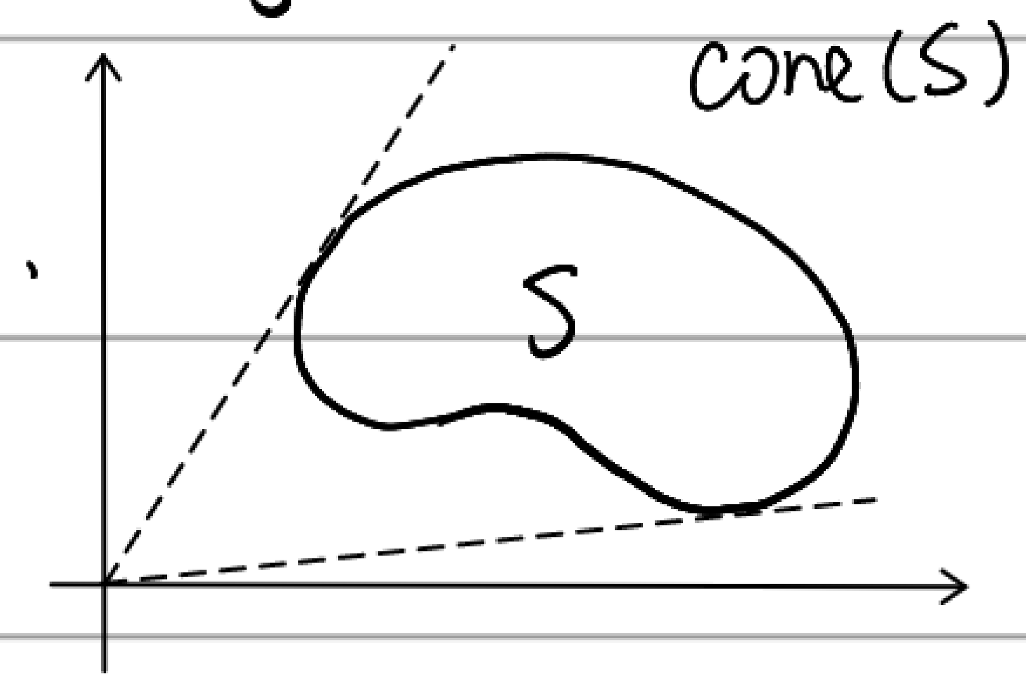
remark: 1 and 2 are called strong alternatives.

weak alternatives: at most one of them can be feasible

def: cone: a set $K \subseteq \mathbb{R}^n$ is a cone if $x \in K \Rightarrow \theta x \in K$ for $\forall \theta \geq 0$.

def: conic hull: given a set S. the conic hull of S is given by

$$\text{cone}(S) \triangleq \left\{ \sum_{i=1}^{n} \theta_i x_i : \theta_i \geq 0.\ x_i \in S \right\}.$$



where $\sum_{i=1}^{n} \theta_i x_i$ is called conic combination.

conic hulls are convex sets.

Proof of Farkas lemma: let $a_1 \cdots a_n$ be colomns of A.

$C = \text{cone}\{a_1 \cdots a_n\}$ is convex. then.

strictly.

$b \notin C \iff \exists$ SH separating C and b.

Problem: is C closed? counterexample $\nearrow$