Lecture 9. Linear programming.

Linear program: $\min_x \; c^T x \quad \text{s.t. } Bx \le d. \; Ax = b.$

Standard form: $\min_x \; c^T x \quad \text{s.t. } Ax = b. \; x \ge 0.$

- adding slack variables. s. $\min_{x,s} \; c^T x. \; \text{s.t. } Bx + s = d. \; s \ge 0.$
- splitting variables into positive and negative parts $x = x^+ - x^-$.

$$\min_{x^+, x^-, s} \; c^T x^+ - c^T x^- \quad \text{s.t. } Bx^+ - Bx^- + s = d. \; Ax^+ - Ax^- = b. \; x^+, x^-, s \ge 0$$

Example. $\min -2x_1 - 3x_2 \quad \text{s.t. } x_1 \le 100. \; x_2 \le 200. \; x_1 + x_2 \le 160.$

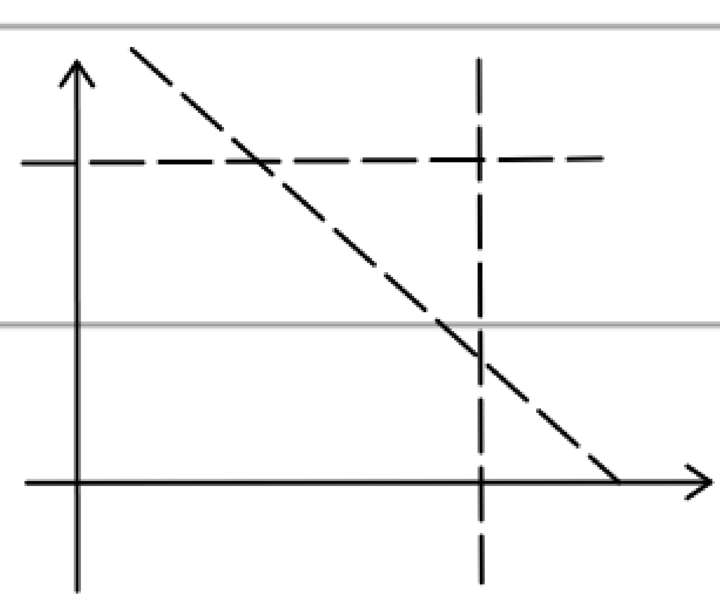
step 1. $\min -2x_1 - 3x_2. \quad \text{s.t. } x_1 + s_1 = 100. \; x_2 + s_2 = 200 \; x_1 + x_2 + s_3 = 160.$

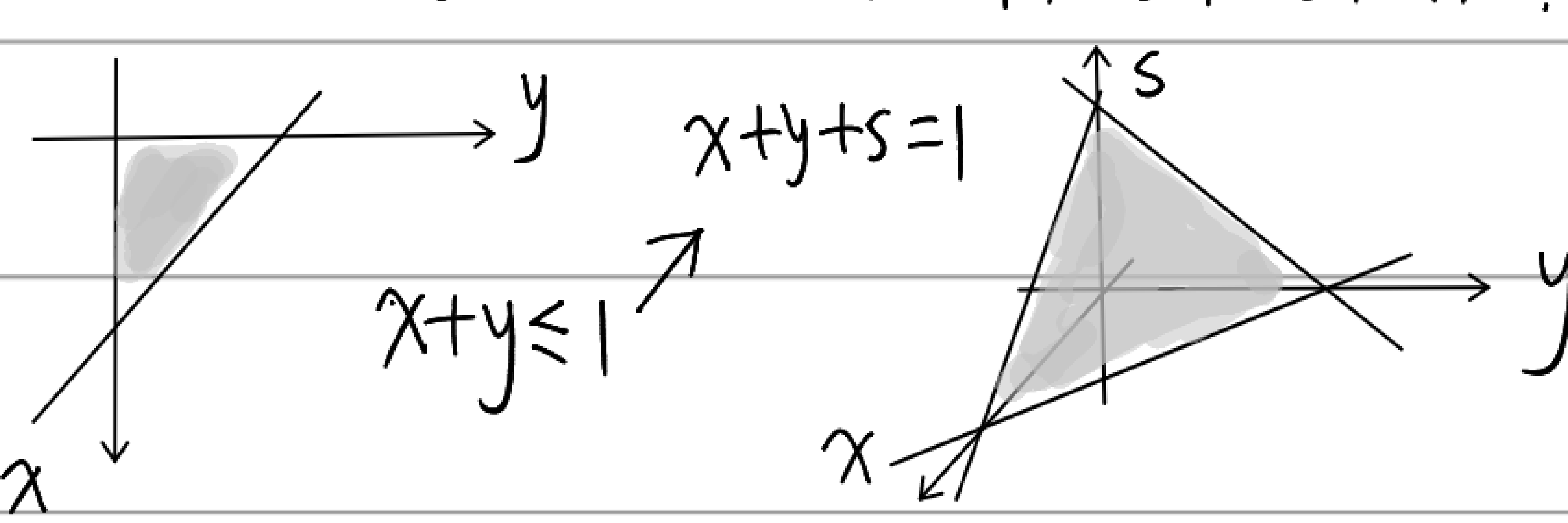
$s_1, s_2, s_3 \ge 0.$

step 2. $\min -2(x_1^+ - x_1^-) - 3(x_2^+ - x_2^-)$

$\text{s.t. } (x_1^+ - x_1^-) + s_1 = 100, \; (x_2^+ - x_2^-) + s_2 = 200.$

$(x_1^+ - x_1^-) + (x_2^+ - x_2^-) + s_3 = 160. \quad x_1^+, x_1^-, x_2^+, x_2^-, s_1, s_2, s_3 \ge 0.$





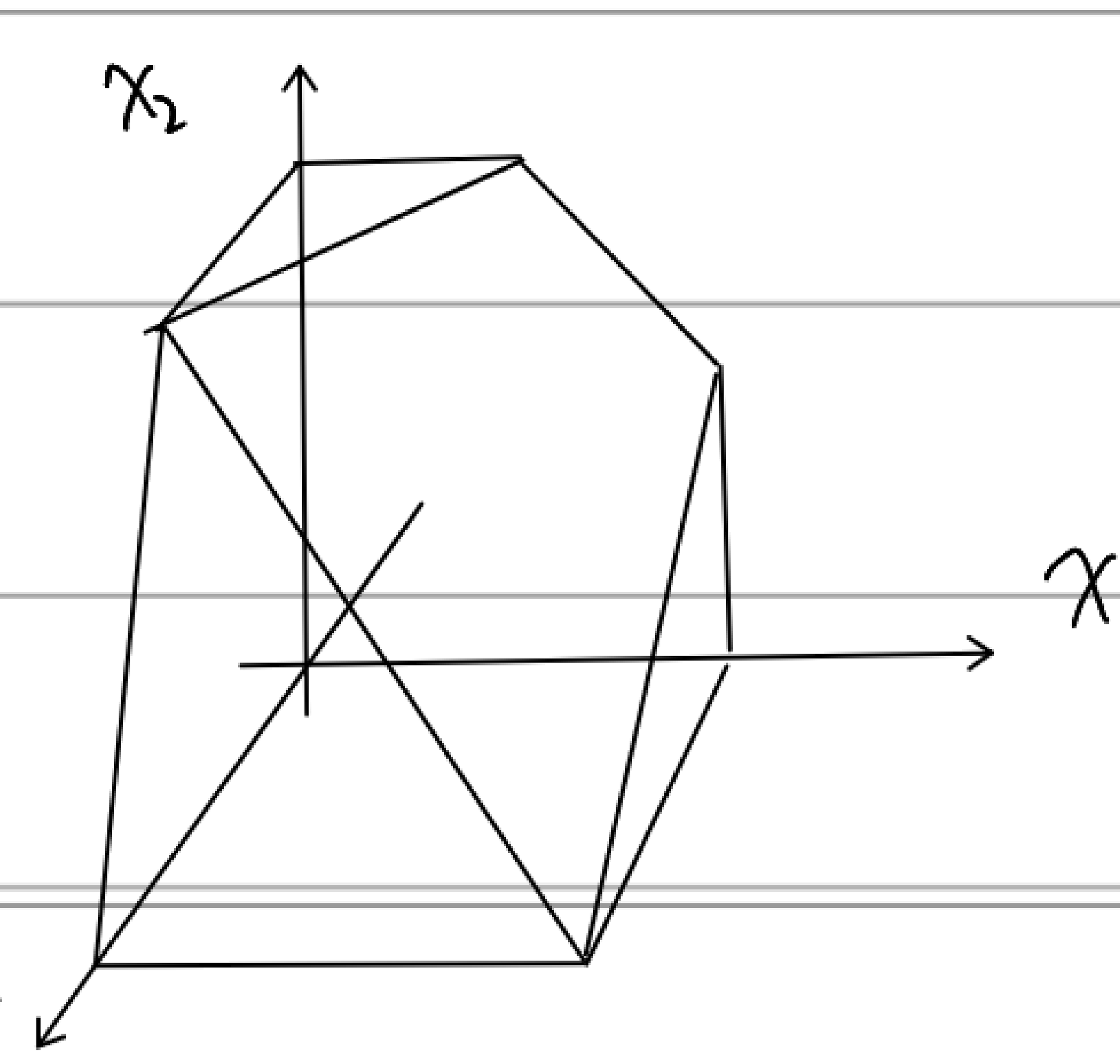
Example. $\min -x_1 + 6x_2 - 13x_3$

$x_1 + x_2 + x_3 \le 400$

$x_2 + 3x_3 \le 600.$

$x_1 \le 200, \; x_2 \le 300$

$x_1, x_2, x_3 \ge 0.$



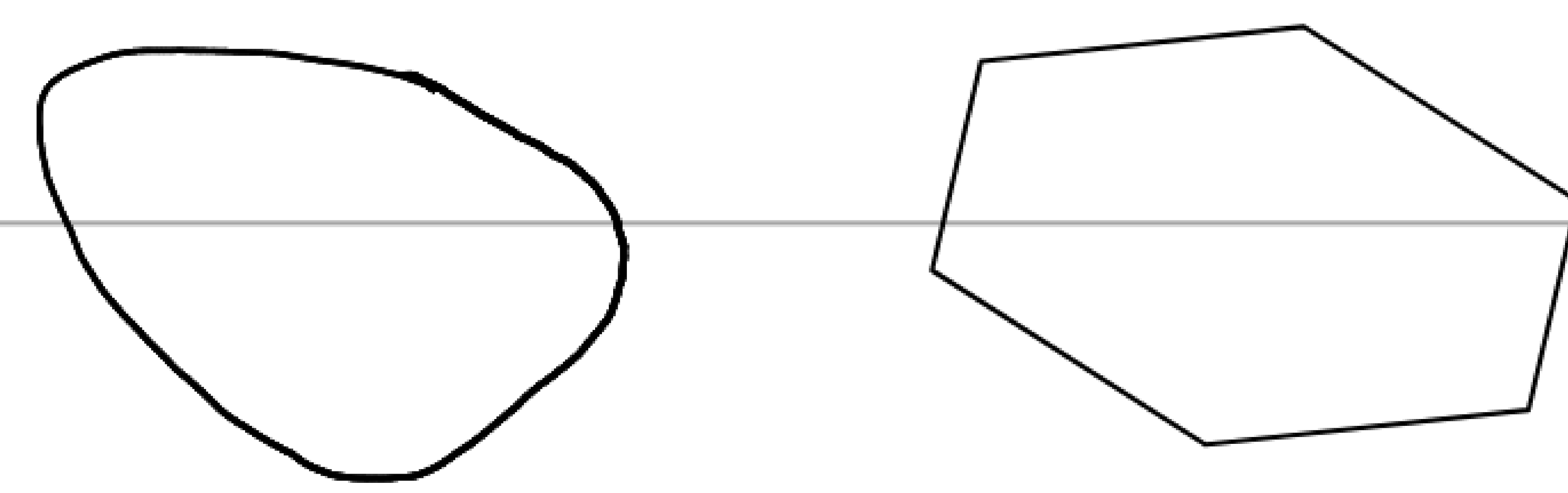
All possibilities for an L.P. infeasible / unbounded / $\exists$ optimal solutions

optimal : infinite number of optimal solutions / unique optimal solution

Conjecture : $\exists$ an optimal solution at a vertex if there exists optimal.

Extreme points : $x \in C$ (convex) is an extreme point if x is not a convex combination of two other points. i.e. $\nexists\ \theta, y, z$ s.t. $x = \theta y + \bar{\theta} z$, $y \neq z$.

what is / are extreme points?



Vertex : $x \in \mathbb{R}^n$ is a vertex of polyhedron P if $x \in P$ and $\exists\, n$ linear independent constraints that are tight at x.

Basic solution : $Ax = b$. $A \in \mathbb{R}^{m\times n}$. $\exists\, m$ linearly independent columns.

$A = (\underset{}{\overset{\in \mathbb{R}^{m\times m}}{B}}, D)$. $x = (x_B^T, 0^T)^T$ basic solution. $x \geq 0$. basic feasible sol.

Another view by slackness. n variables. m constraints n slack variable.

$Ax \leq b \rightarrow A'x' = b$ $\quad \mathbb{R}^{m\times n} \rightarrow \mathbb{R}^{m\times(n+m)}$ $\quad x_B \in \mathbb{R}^m$. $0 \in \mathbb{R}^n$

vertex : n tight constraints $\Rightarrow$ n slack variable $= 0$

Equivalence of extreme point and vertex :

"$\Leftarrow$". $x \in P$ be a vertex. $\Rightarrow$ n linear independent constraints tight at x.

$\Rightarrow \exists \tilde{A} \in \mathbb{R}^{n\times n}, \tilde{b} \in \mathbb{R}^n$. s.t. $\tilde{A}x = \tilde{b}$. Suppose $x = \theta y + \bar{\theta} z$. $y, z \in P$

$\Rightarrow \theta \tilde{A} y + \bar{\theta} \tilde{A} z = \tilde{b}$. $\tilde{A}y, \tilde{A}z \le b \Rightarrow \tilde{A}y = \tilde{A}z = b$. but $\tilde{A}$ invertible.

"$\Rightarrow$". suppose $x \in P$ not a vertex. Let $I = \{i : a_i^T x = b_i\}$.

so there does not exist n linearly independent a_i s.t. $i \in I$.

$a_i \in \mathbb{R}^n \Rightarrow \exists \underset{\ne 0}{d} \in \mathbb{R}^n$, $d^T a_i = 0$. $\forall i \in I$. let $y = x + \varepsilon d$, $z = x - \varepsilon d$.

$\forall i \in I$. $a_i^T y = a_i^T z = b$. since $a_i^T d = 0$. $\forall i \notin I$. $f(w) = b_i - a_i^T w$

continuous. and $f(x) > 0$. choose ε sufficiently small. $f(y)$. $f(z) \ge 0$ $\square$

Corollary. given a finite set of linear inequalities, only finite extreme points.

Existence of extreme points: iff P does not contain a line.

Corollary: every bounded polyhedron (every polytope) has an extreme.

Fundamental theorem of linear programming.

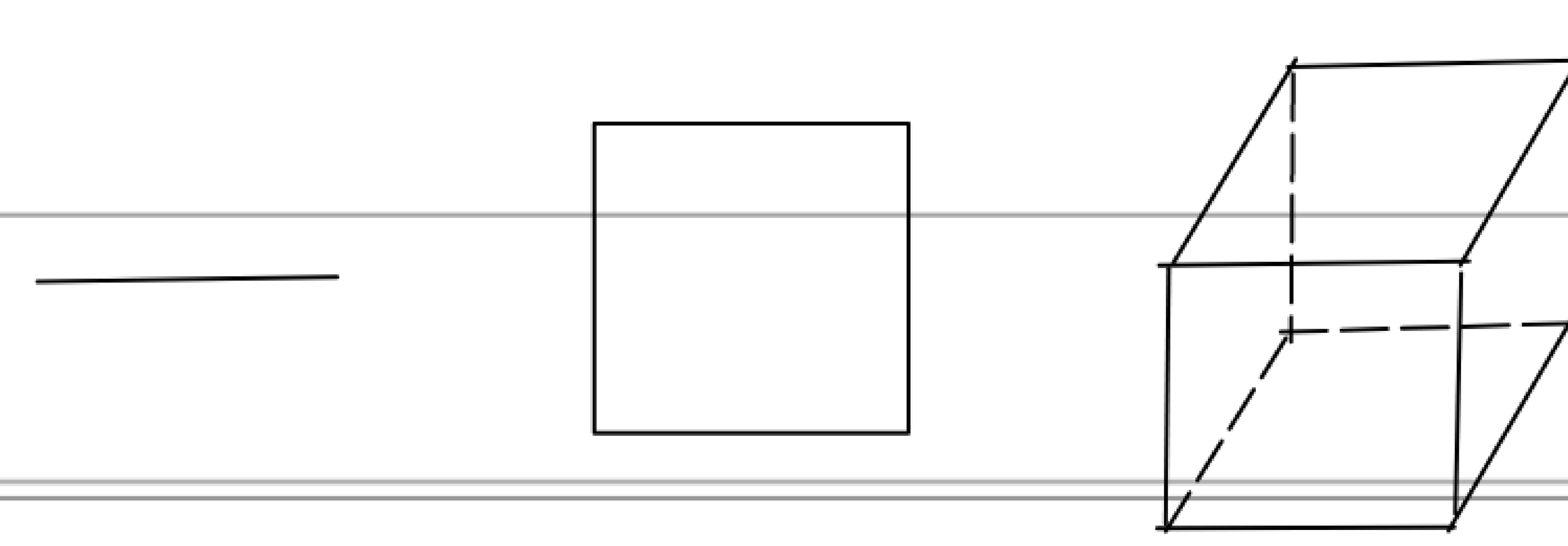
consider the linear program. $\min c^T x$. s.t. $Ax \le b$. $x \ge 0$

suppose P has ≥ 1 optimal solution. then $\exists$ an optimal sol at a vertex.

Algorithmic application: find optimal solution by enumerating $\binom{m}{n}$ vertices.

However. $\{0,1\}^n$ - cube:

$2n$ inequalities.
but 2^n vertices.



Proof: Let Q be the set of optimal solutions. $Q \neq \phi$.

P has an extreme point $\Rightarrow$ P has no lines $\Rightarrow Q \subseteq P$ has no lines

$\Rightarrow Q$ has an extreme point x^*. We now show that x^* is extreme of P.

Let $c^T x^* = v$. and suppose $x^* = \theta x_1 + \bar{\theta} x_2$. $x_1, x_2 \in P$. $\theta \in (0,1)$.

$\Rightarrow v = \theta c^T x_1 + \bar{\theta} c^T x_2$. $x^* \in Q$ is an optimal solution of P.

$\Rightarrow c^T x_1, c^T x_2 \leq v$. $\Rightarrow c^T x_1 = c^T x_2 = v \Rightarrow x_1, x_2 \in Q$. contradiction. $\square$

Simplex method: ① start at a vertex. ② if $\exists$ better neighbour, move to it.

Definition: two vertices are neighbours if share $n-1$ tight constraints.

start from 0. $\min c^T x$. if $c_i < 0$ x_i should increase

increase x_i to make another constraint tight. then shift coordinate.

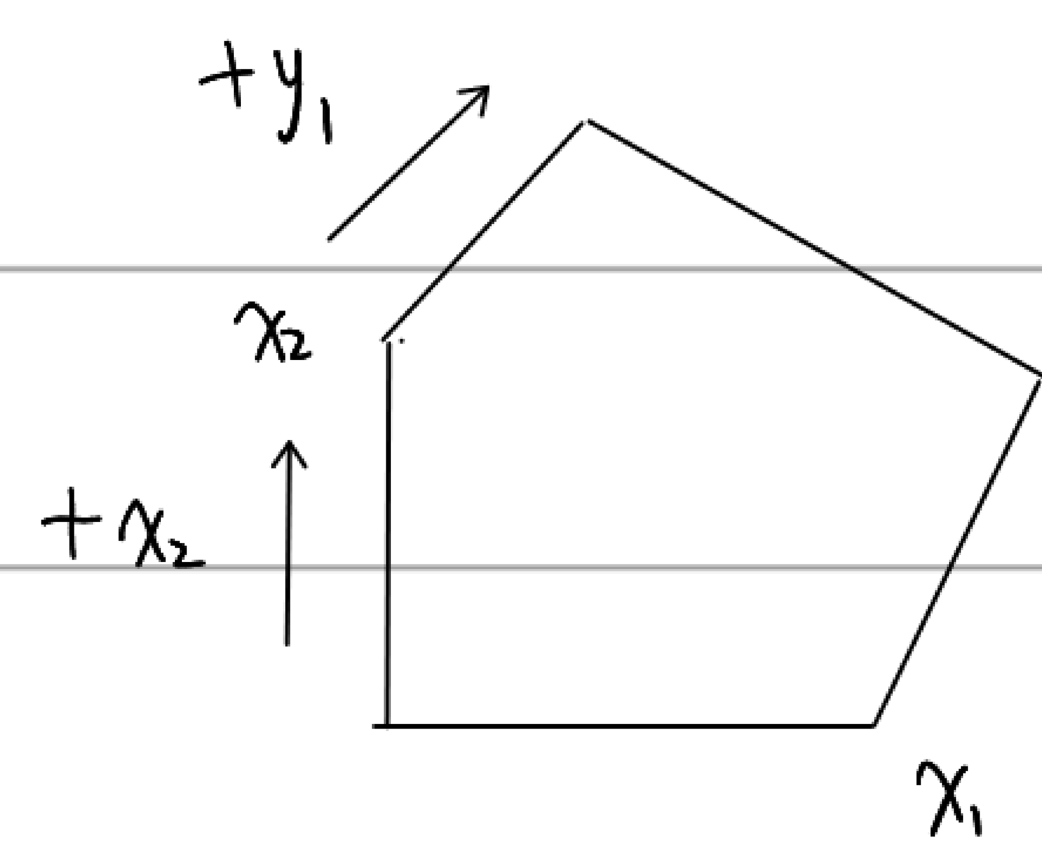
$\min -2x_1 - 5x_2$

s.t. $2x_1 - x_2 \leq 4$

$x_1 + 2x_2 \leq 9$

$-x_1 + x_2 \leq 3$

$x_1, x_2 \geq 0$.



initial LP

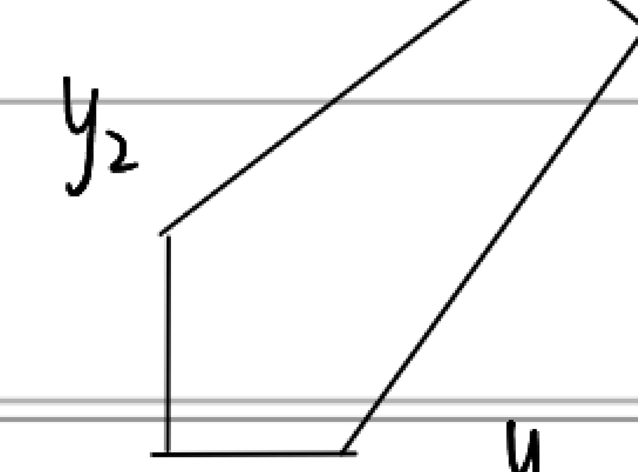
start from $(0,0)$

increase x_2 to make $-x_1 + x_2 \leq 3$ tight. $y_1 = x_1$. $y_2 = 3 + x_1 - x_2$

$\min -15 - 7y_1 + 5y_2$

s.t. $y_1 + y_2 \leq 7$. $-y_1 + y_2 \leq 3$.

$3y_1 - 2y_2 \leq 3$. $y_1, y_2 \geq 0$.



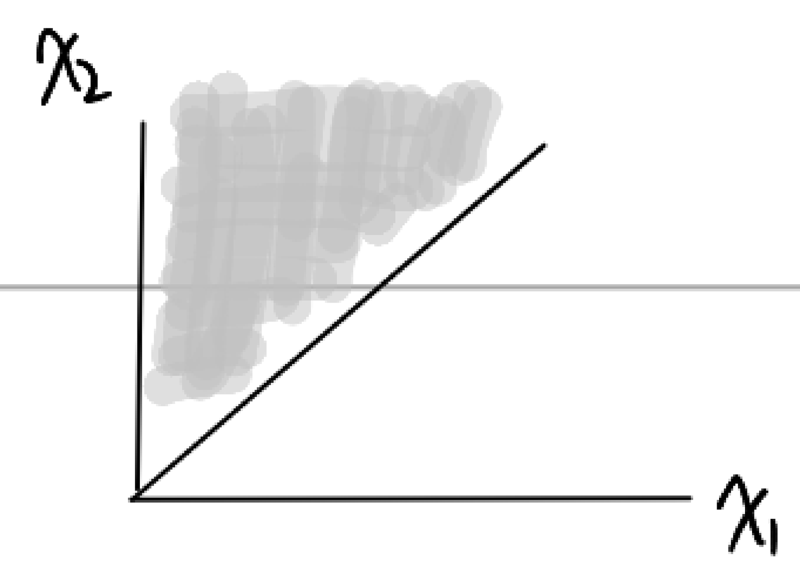
increase y_1 (note: fix $y_2 = 3 + x_1 - x_2 = 0$). to make $3y_1 - 2y_2 \le 3$ tight.

now $y_1 = 1$. let $z_1 = 3 - 3y_1 + 2y_2$, $z_2 = y_2$. rewrite LP.

min $-22 + \frac{7}{3} z_1 + \frac{1}{3} z_2$. $-\frac{1}{3} z_1 + \frac{5}{3} z_2 \le 6$. $z_1 \geq 0$. $z_2 \geq 0$, ---

Degeneracy. 退化. more than n constraints tight at a vertex.

$x_1 - x_2 \le 0$
$x_1, x_2 \geq 0$.



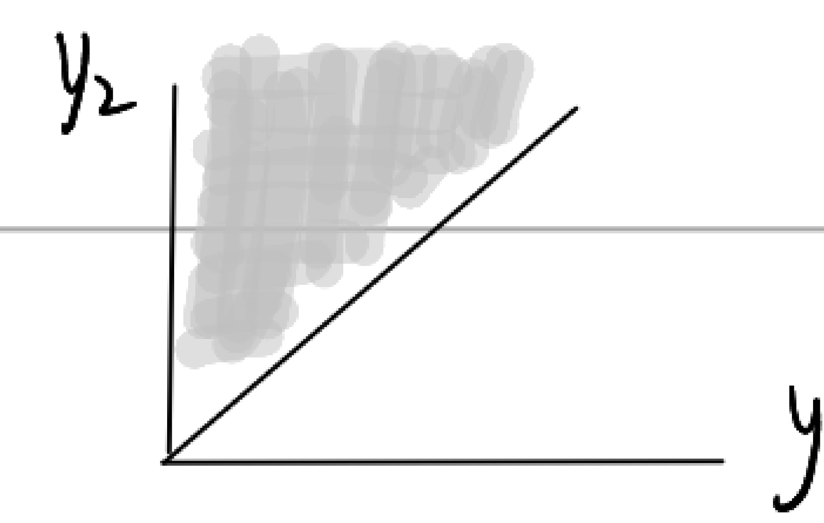
increase x_1

make $x_1 - x_2 \le 0$ tight. $x_1 = 0$.

$y_1 = x_2 - x_1$, $y_2 = x_2$

$y_1 \geq 0$ $y_2 \geq 0$
$y_1 - y_2 \le 0$.

increase y_1



fail!!!

how to fix degeneracy? break cycles / add perturbation $b_i' = b_i \pm \varepsilon_i$