CS-3334: Advanced Combinatorics **Fall 2022**

Lecture 6: October 18

Lecturer: Kuan Yang *Scribe: Weihao Zhu*

We have showed that surprisingly many tempting conjectures can be easily disproves by the probabilistic method and random graphs. Today, we will introduce threshold functions of random graphs.

6.1 Graph Property & Threshold Functions

Definition 6.1 *A graph property $\mathcal{P}$ is a subset of all graphs.*

We say a graph property $\mathcal{P}$ is monotone increasing/decreasing if for any $G \in \mathcal{P}$, any graph we obtain through adding/deleting edges in G always belongs to $\mathcal{P}$. For instance, for a fixed graph H, the graph property $\mathcal{P}_1 = \{G : H \text{ is an induced sub-graph of } G\}$ is monotone increasing. The graph property $\mathcal{P}_2 = \{G : G \text{ is a connected planar graph}\}$ is monotone decreasing. However, $\mathcal{P}_3 = \{G : G \text{ contains a vertex of degree } 1\}$ is not monotone.

A graph property $\mathcal{P}$ is non-trivial if for any sufficiently large n, there always exists a graph with n vertices in $\mathcal{P}$ and another graph not in $\mathcal{P}$.

What we want to discuss today is the following problem:

Problem 6.1 *Given a graph property $\mathcal{P}$, for which $p = p_n$ is $\mathcal{P}$ true for $\mathcal{G}(n, p)$ with high probability?*

6.2 Warm-up: Graphs with Triangles

Let's start from the easiest problem. Suppose $\mathcal{P} = \{G : K_3 \subseteq G\}$. Now, consider $G \sim \mathcal{G}(n, p_n)$. Let X be the number of K_3 in graph G, which is a random variable.

If $p \ll \frac{1}{n}$, then $\mathbf{Pr}[X \geq 1] = o(1)$ according to Markov's inequality.

If $p \gg \frac{1}{n}$, let's first prove that $\mathbf{Var}[X] = o(\mathbf{E}[X]^2)$. Denote S as the set of all subsets of vertices in G of size 3, and denote X_T the indicator variable of the set T inducing a triangle in G. Obviously, $X = \sum_{T \in S} X_T$. Notice that

$$\begin{aligned} \mathbf{Cov}[X_{T_1}, X_{T_2}] &= \mathbf{E}[X_{T_1} X_{T_2}] - \mathbf{E}[X_{T_1}] \cdot \mathbf{E}[X_{T_2}] \\ &= p^{|E(T_1 \cup T_2)|} - p^{|E(T_1) + E(T_2)|} \\ &= \begin{cases} 0 & |V(T_1 \cap T_2)| \leq 1 \\ p^5 - p^6 & |V(T_1 \cap T_2)| = 2 \end{cases}. \end{aligned}$$

Also, we have

$$\mathbf{Var}[X_T] = \mathbf{E}[X_T^2] - \mathbf{E}[X_T]^2 = p^3 - p^6.$$

Therefore,

$$\begin{aligned}
\mathbf{Var}[X] &= \sum_{T\in S}\mathbf{Var}[X_T] + \sum_{\substack{T_1,T_2\in S\\ T_1\neq T_2}}\mathbf{Cov}[X_{T_1},X_{T_2}] \\
&= \binom{n}{3}(p^3-p^6) + \sum_{\substack{T_1,T_2\in S\\ T_1\neq T_2\\ |V(T_1\cap T_2)|=2}}(p^5-p^6) \\
&= \binom{n}{3}(p^3-p^6) + \binom{n}{2}(n-2)(n-3)(p^5-p^6) \\
&\lesssim n^3p^3 + n^4p^5 \\
&= o(n^6p^6).
\end{aligned}$$

The last equality above holds as $p \gg \frac{1}{n}$. This implies that $\mathbf{Var}[X] = o(\mathbf{E}[X]^2)$. Based on Chebyshev's inequality, we can see that $\mathbf{Pr}[X=0] = o(1)$.

Here, we give the definition of the threshold function as follows.

Definition 6.2 *We say r_n is a threshold function for some graph property $\mathcal{P}$ if*

$$\mathbf{Pr}[\mathcal{G}(n,p_n)\in\mathcal{P}] \to \begin{cases} 0 & \text{if } p_n/r_n \to 0 \\ 1 & \text{if } p_n/r_n \to \infty \end{cases}.$$

From above, we are able to come to the following theorem.

Theorem 6.1 *A threshold function for containing a K_3 is $\frac{1}{n}$.*

6.3 Threshold Function for Containing A Given Graph

In course *Advanced Algorithms*, we have already known that a threshold function for containing a K_4 is $n^{-2/3}$. We now consider some general cases.

Suppose we have a random variable $X = X_1 + \ldots + X_m$, where X_i is the indicator of event E_i. We say $i \sim j$ is $i \neq j$ and E_i, E_j are not independent. If $i \neq j$ and $i \not\sim j$, we clearly have $\mathbf{Cov}[X_i, X_j] = 0$. Otherwise,

$$\mathbf{Cov}[X_i,X_j] = \mathbf{E}[X_iX_j] - \mathbf{E}[X_i]\mathbf{E}[X_j] \le \mathbf{E}[X_iX_j] = \mathbf{Pr}[E_i \wedge E_j].$$

Also note that $\mathbf{Var}[X_i] \le \mathbf{E}[X_i^2] = \mathbf{E}[X_i]$, which implies that

$$\mathbf{Var}[X] \le \mathbf{E}[X] + \sum_{i\sim j}\mathbf{Pr}[E_i \wedge E_j].$$

Define $\Delta := \sum_{i\sim j}\mathbf{Pr}[E_i \wedge E_j]$. We hope $\mathbf{Var}[X] = o(\mathbf{E}[X])^2$, so if $\mathbf{E}[X] \to \infty$, $\Delta = o(\mathbf{E}[X])^2$ suffices. Moreover,

$$\sum_{i\sim j}\mathbf{Pr}[E_i \wedge E_j] = \sum_i \mathbf{Pr}[E_i]\sum_{j\sim i}\mathbf{Pr}[E_j|E_i].$$

In many symmetric cases, $\sum_{j\sim i}\mathbf{Pr}[E_j|E_i]$ does not depend on i. Denote it by Δ^* (or we may set $\Delta^* = \max_i \sum_{j\sim i}\mathbf{Pr}[E_j|E_i]$ in asymmetric cases). Therefore, $\Delta = \sum_i \mathbf{Pr}[E_i]\Delta^* = \mathbf{E}[X]\Delta^*$. This gives us the following lemma.

Lemma 6.2 *If $\mathbf{E}[X] \to \infty$ and $\Delta^* = o(\mathbf{E}[X])$, then $X > 0$ with high probability.*

In fact, by Chebyshev's inequality, we have

$$\Pr[(1-\varepsilon)\mathbf{E}[X] \le X \le (1+\varepsilon)\mathbf{E}[X]] \ge 1 - \frac{\mathbf{Var}[X]}{\varepsilon^2 \mathbf{E}[X]^2} = 1 - o(1)$$

for any constant $0 < \varepsilon < 1$.

Now consider the property of containing K_4. For any set S consisting of exactly four vertices, let E_S be the event that S forms a K_4 in the random graph. For any S, T of size 4, $S \sim T$ if and only if $|S \cap T| \geq 2$. There are two cases:

- $|S \cap T| = 2$:
$$\sum_T \mathbf{Pr}[E_T | E_S] \le 6\binom{n}{2}\mathbf{Pr}[E_T|E_S] = 6\binom{n}{2}p^5 \approx n^2p^5;$$
- $|S \cap T| = 3$:
$$\sum_T \mathbf{Pr}[E_T|E_S] = 4(n-4)\mathbf{Pr}[E_T|E_S] \le 4np^3 \approx np^3.$$

Therefore, $\Delta^* \approx n^2p^5 + np^3 = o(n^4p^6) = o(\mathbf{E}[X])$ if $n^2p \gg 1$ and $np \gg 1$.

One may ask letting X be the number of a general graph H, can we still say that $X > 0$ with high probability if $\mathbf{E}[X] \to \infty$? This is actually not correct. Suppose H is the graph as follows (obtained by adding an edge to K_4). Then, $\mathbf{E}[X] \approx n^5p^7 \to \infty$ if $p \gg n^{-5/7}$. However, there is no K_4 in $\mathcal{G}(n,p)$ if $p \ll n^{-2/3}$.

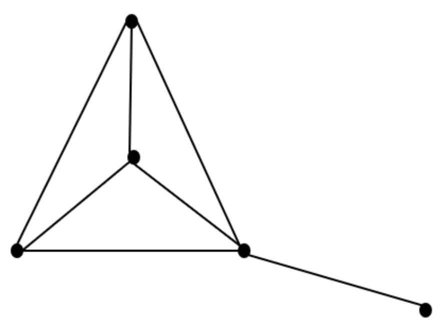

Figure 6.1: An counterexample of the conjecture above.

So, can we find a threshold function for containing a general graph? The following theorem tells us the answer.

Definition 6.3 *The edge-vertex ratio of $G = (V, E)$ is defined as $\rho(G) = |E|/|V|$. The maximum sub-graph ratio is given by $m(G) = \max_{H \subseteq G} \rho(H)$.*

Theorem 6.3 (Bollobás, 1981) *Fix a graph $H = (V, E)$. Then $p = n^{-1/m(H)}$ is a threshold function for containing H as a sub-graph. Furthermore, if $p \gg n^{-1/m(H)}$, then X_H (number of copies of H in $\mathcal{G}(n,p)$) with high probability satisfies*

$$X_H \approx \mathbf{E}[X] = \binom{n}{|V|}\frac{|V|!}{|Aut(H)|}p^{|E|} \approx \frac{n^{|V|}p^{|E|}}{|Aut(H)|}.$$

Proof: Let H' be the sub-graph of H achieving the maximum edge-vertex ratio, i.e., $m(H) = \rho(H')$. If $p \ll n^{-1/m(H)}$, then $\mathbf{E}[X_{H'}] = o(1)$, which implies that $X_{H'} = 0$ with high probability.

Now assume that $p \gg n^{-1/m(H)}$. Count the labelled copies of H in $\mathcal{G}(n,p)$. Let L be a labelled copy of H in K_n. A_L be the event of $L \subseteq \mathcal{G}(n,p)$. For fixed L, we have

$$\Delta^* = \sum_{L' \sim L} \mathbf{Pr}[A_{L'}|A_L] = \sum_{L' \sim L} p^{|E(L') \setminus E(L)|}.$$

Note that the number of L' such that $L' \sim L$ is approximately $n^{|V(L') \setminus V(L)|}$, and

$$p \gg n^{-1/m(H)} \gg n^{-1/\rho(L' \cap L)} = n^{-|V(L') \cap V(L)|/|E(L') \cap E(L)|}.$$

So, we have

$$\Delta^* \approx \sum n^{|V(L') \setminus V(L)|} p^{|E(L') \setminus E(L)|} \ll n^{|V(L)|} p^{|E(L)|},$$

which implies that $\Delta^* \ll \mathbf{E}[X_H]$. Therefore, $\mathbf{Var}[X] = \mathbf{E}[X_H] + o(\mathbf{E}[X_H])^2$, which completes the proof. ■

6.4 Existence of Threshold

In this section, we consider for which graph property $\mathcal{P}$ does a threshold function exist?

Let's start from a simpler question. Assume that $\mathcal{P}$ is monotone increasing, is $f(p) = \mathbf{Pr}[\mathcal{G}(n,p) \in \mathcal{P}]$ increasing? We first discuss the question on upward closed sets.

Let $\mathcal{F}$ be a family of subsets of $[n]$. We call $\mathcal{F}$ an upward closed set (or up-set) if for any $S \subseteq T$ and $S \in \mathcal{F}$, we have $T \in \mathcal{F}$. We have the following theorem.

Theorem 6.4 *Suppose $\mathcal{F}$ is a non-trivial ($\mathcal{F} \neq \emptyset$ or $2^{[n]}$) up-set of $[n]$. Let $Bin([n],p)$ be a random set where each number in $[n]$ is chosen independently with probability p. Then $f(P) = \mathbf{Pr}[Bin([n],p) \in \mathcal{F}]$ is a strictly increasing function.*

Proof: We prove it by *coupling*. For any $0 \leq p < q < 1$, construct a coupling as follows. Pick a uniform random vector $(x_1, \dots, x_n) \in [0,1]^n$. Let $A = \{i : x_i \leq p\}$ and $B = \{j : x_j \leq q\}$. Clearly, A has the same distribution as $\text{Bin}([n],p)$ and B has the same distribution as $\text{Bin}([n],q)$. Notice that $A \subseteq B$. Thus, we have

$$f(p) = \mathbf{Pr}[A \in \mathcal{F}] < \mathbf{Pr}[B \in \mathcal{F}] = f(q),$$

which completes the proof. ■

Here, we give another proof, which is based on two-round exposure coupling.

Proof: Let $0 \leq p < q \leq 1$. Construct A, B as follows:

- For any $i \in [n]$, add i into A with probability p.
- If $i \in A$, add i into B. Otherwise, add it into B with probability $1 - \frac{1-q}{1-p}$.

Notice that $\mathbf{Pr}[i \in B] = p + (1-p) \cdot (1 - \frac{1-q}{1-p}) = q$. Therefore, A has the same distribution as $\text{Bin}([n],p)$ and B has the same distribution as $\text{Bin}([n],q)$. The rest of the proof is the same. ■

Now, let's prove that every non-trivial monotone increasing graph property has a threshold function.

Theorem 6.5 (Bollobás & Thomason, 1987) *Every non-trivial monotone increasing graph property has a threshold function.*

Proof: Consider k independent copies $G_1, G_2, \ldots, G_k$ of $\mathcal{G}(n,p)$. Their union $G_1 \cup \ldots \cup G_k$ has the same distribution of $\mathcal{G}(n, 1-(1-p)^k)$. According to the monotonicity of $\mathcal{P}$, if $G_1 \cup \ldots \cup G_k \notin \mathcal{P}$, then $G_i \notin \mathcal{P}$ for all $1 \le i \le k$. Note that these k copies are independent, we have

$$\mathbf{Pr}[\mathcal{G}(n, 1-(1-p)^k) \notin \mathcal{P}] \le \mathbf{Pr}[\mathcal{G}(n,p) \notin \mathcal{P}]^k.$$

Let $f(p) = f_n(p) = \mathbf{Pr}[\mathcal{G}(n,p) \in \mathcal{P}]$. Note that $(1-p)^k \ge 1 - kp$. For any monotone increasing property $\mathcal{P}$ and any positive integer $k \le \frac{1}{p}$, we have

$$1 - f(kp) \le 1 - f(1-(1-p)^k) \le (1-f(p))^k.$$

For any sufficiently large n, define a function as follows. Since $f(p)$ is a continuous strictly increasing function from 0 to 1 as p goes from 0 to 1, there is some critical $p_c = p_c(n)$ such that $f(p_c) = \frac{1}{2}$. We claim that p_c is a threshold function.

If $p = p(n) \gg p_c$, then letting $k = \lceil p/p_c \rceil \to \infty$, we have $1 - f(p) \le (1 - f(p_c))^k = 2^{-k} \to 0$. Therefore, $f(p) \to 1$.

Analogously, if $p \ll p_c$, then letting $\ell = \lceil p/p_c \rceil \to \infty$, we have $\frac{1}{2} = 1 - f(p_c) \le (1-f(p))^\ell$. Thus, $f(p) \to 0$ as $n \to \infty$. This completes the proof. ■

6.5 Sharp Threshold

In fact, using the method of moments, the number of triangles in a random graph converges to a Poisson distribution. We have

$$\mathbf{Pr}[\text{A triangle exists in } \mathcal{G}(n, c_n/n)] \to \begin{cases} 0 & \text{if } c_n \to -\infty \\ 1 - e^{-c^3/6} & \text{if } c_n \to c \\ 1 & \text{if } c_n \to \infty \end{cases}.$$

However, consider some other properties, such as "no isolated vertex". We have

$$\mathbf{Pr}[\mathcal{G}(n,p) \text{ has no isolated vertex}] = e^{-e^{-c}}$$

if $c_n \to c$, where $p = \frac{\log n + c_n}{n}$ and $c \in R \cup \{-\infty, \infty\}$. (We leave it as an exercise.) Note that if $c_n \to -\infty$, even though $c_n = -o(\log n)$, we have the probability goes to $e^{-e^{-c}} = 0$. Analogously, $e^{-e^{-c}} = 1$ if $c_n \to \infty$, even though $c_n = o(\log n)$. So this property shows a stronger notion of threshold: sharp threshold.

Definition 6.4 *We say r_n is a sharp threshold for some graph property $\mathcal{P}$ if for any $\delta > 0$, we have*

$$\mathbf{Pr}[\mathcal{G}(n,p_n) \in \mathcal{P}] \to \begin{cases} 0 & \textit{if } p_n \le (1-\delta) r_n \\ 1 & \textit{if } p_n \ge (1+\delta) r_n \end{cases}.$$

Roughly speaking, any monotone graph property with a coarse threshold may be approximated by a local property (having some H as a sub-graph). This is the famous Friedgut's sharp threshold theorem, which was proved in 1999.

A well-known conjecture is if the property of not being k-colorable has a sharp threshold for some constant (only depending on k) threshold d_k. Namely, we are interested in whether a constant d_k exists, such that

$$\mathbf{Pr}[\mathcal{G}(n,p_n) \text{ is } k\text{-colorable}] \to \begin{cases} 1 & \text{if } d(n) < d_k \\ 0 & \text{if } d(n) > d_k \end{cases} .$$

The following theorem shows that the property of being k-colorable indeed has a sharp threshold.

Theorem 6.6 (Achlioptas & Friedgut, 2000) *For any $k \geq 3$, there exists a function $d_k(n)$ such that for any $\varepsilon > 0$, we have*

$$\mathbf{Pr}[\mathcal{G}(n,p_n) \text{ is } k\text{-colorable}] \to \begin{cases} 1 & d(n) < d_k(n) - \varepsilon \\ 0 & d(n) > d_k(n) + \varepsilon \end{cases} .$$

However, it still remains an open question whether $d_k(n)$ has a limit d_k.

6.6 Clique number and chromatic number of $\mathcal{G}(n, 1/2)$

We now consider an easier case: the chromatic number of $\mathcal{G}(n,1/2)$ instead. As we have known in course Advanced Algorithms, it has a strong concentration on its expectation. Now we would like to compute its expectation.

Note that $\mathcal{G}(n,1/2)$ has the same distribution of its complement. So we have $\omega(\mathcal{G}(n,1/2)) = \alpha(\mathcal{G}(n,1/2))$. It is also well-known that $\chi(G) \geq |V(G)|/\alpha(G)$. We first compute the clique number of $\mathcal{G}(n,1/2)$.

Let X be the number of k-cliques in $\mathcal{G}(n,1/2)$. Then we have

$$\mathbf{E}[X] = \binom{n}{k} 2^{-\binom{k}{2}}.$$

Denote it by $f(k)$. Clearly $\omega < k$ if $f(k) \to 0$. Now assume $f(k) \to \infty$. Let A_S be the event that S forms a clique in $\mathcal{G}(n,1/2)$. Fix S,T of size k. Then $S \sim T$ if $|S \cap T| \geq 2$. So we have

$$\Delta^* = \sum_{T \sim S} \Pr[A_T \mid A_S] = \sum_{\ell=2}^{k-1} \binom{k}{\ell} \binom{n-k}{k-\ell} 2^{\binom{\ell}{2} - \binom{k}{2}} .$$

We claim that $\Delta^* = o(f(k))$ if $f(k) \to \infty$ (details are omitted temporarily). Thus we have $X > 0$ (i.e., $\omega \geq k$) with high probability.

Theorem 6.7

$$\omega(\mathcal{G}(n,1/2)) \approx 2\log_2 n .$$

This theorem yields the following corollary immediately.

Lemma 6.8

$$\chi(\mathcal{G}(n,1/2)) \geq \frac{n}{\alpha(\mathcal{G}(n,1/2))} = \frac{n}{\alpha(\mathcal{G}(n,1/2))} \geq (1-o(1)) \frac{n}{2\log_2 n} .$$

However, how can we upper bound the chromatic number?

(To be continued...)